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AN EMPIRICAL EVALUATION OF LATENT CLASS ANALYSIS

by

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ABSTRACT

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The purpose of this project is to empirically evaluate the theoretical model of latent class analysis proposed by T. W. Anderson (Psychometrika, 1954, 19, 1-10). This model was designed to determine if a population of respondents could be divided into a finite number of m distinct groups or classes. A basic assumption of the latent class model is that the responses of subjects in the same class to different dichotomous items are statistically independent. Using the manifest probabilities of positive response of the persons in a random sample, the computational method provides estimates of the latent parameters, that is, the proportion of the population in each latent class and the probability of positive response associated with each item for each latent class.

A history of the development of this topic is covered; this includes a detailed description of the computational procedures of B. F. Green, Jr. (Psychometrika, 1951, 16, 151-166) and Anderson. After a modification of Anderson's is presented, the model is tested using a population of known characteristics. The properties of the solution are described and an evaluation of the method is given.

PREFACE

One goal associated with the application of mathematical terminology and techniques to psychological data is the introduction of structure into a seemingly unordered domain. As an example, factor analysis is the generic name for many different methods of analyzing the intercorrelations between a set of variables such as tests or even items within a test. Whether the purpose is to evaluate a hypothesis concerning the nature of mental ability or, as in most instances, to determine the minimum number of independent dimensions which are necessary to explain most of the variance in the original variables, the result is a structuring of the data. This leads to a clearer and more accurate comprehension of the underlying factors in a given situation than could have been attained on the basis of the initial observations alone.

Although factor analysis is the best known, it is not the only structural method to be considered for the purpose of establishing hidden meaning in observed behavior. From research on attitude measurement, Lazarsfeld [1950] proposed the latent structure model, a means of determining the probability of choosing a specific alternative of a dichotomous item for those respondents in the population having the same attitude strength. Utilizing coordinate axes, the attitude continuum was represented by the abscissa and the probability of response by the ordinate; in this way, a probability trace line for each item became hypothetically defined. In a practical situation, this model was designed to provide estimates of the trace line parameters from the responses of randomly sampled subjects.

Green [1951] and Anderson [1954] modified Lazarsfeld's approach to one which was called the latent class model. By making the additional assumption that the population of respondents could be divided into a finite number of discrete classes, they eliminated the problem of postulating a specific relationship between the trait of concern and the response probability such as, for example, a linear trace line model. McHugh [1956] and Gibson [1955, 1962] extended Anderson's solution in order to improve the resulting probability estimates. Because of its greater range of possible application, much more attention has been devoted to the latent class model than to its predecessor, the latent structure model.

It is surprising to note that, to this date, the latent class model has received only theoretical consideration in psychological research. Gibson [1959], in commenting on its potential usefulness, has shown that this model avoids many of the problems inherent in the factor analytic model, namely communality estimation, rotation, and curvilinearity. The major difficulty in any practical use of this technique would be its computational complexity. Eliminating this barrier by taking advantage of modern electronic data-processing methods, the goals of this research are to assess the worth of the latent class model by empirical means and to give recommendations regarding its future application.

The selection of this topic as a thesis project occurred quite by accident. As a graduate student in the Department of Psychology of the University of Maryland, the author was assigned to present an article on a proposed computational model published in Psychometrika by T. W. Anderson. With only a very basic knowledge of matrix algebra, the initial reading of this paper was, to say the least, disconcerting. However, after much effort, the article was finally presented although its use as the basis of a master's thesis had yet to be contemplated. Interest was only aroused when a subsequent search of the professional journals revealed the lack of empirical research on this model. Perhaps the greatest impetus to further investigation was the potential usefulness of this model as a classification technique. Anderson's proposal avoided difficulties inherent in the models of Lazarsfeld and Green and an evaluation of it seemed long overdue. Specifically, as this is one of many untested models in the psychological literature, this study is aimed at the solution of a mystery: Does the latent class model advanced by Anderson do what it has been designed to do given certain conditions and, if so, how well? Needless to say, a rigorous answer to this question necessitates the use of the computer, a tool which is undoubtedly becoming increasingly more important in the testing and reconstruction of existing models and theories as well as in the formulation of new ones.

The first step to be taken was the development of a computer program to carry out the specified computations. This task was initiated in the summer of 1964; the final version of this program (LSA5) was completed in December of the same year. In addition, two other programs associated with this project were developed in February of 1965. The first of these (THEO) is essentially a matrix multiplication program to provide the theoretical manifest matrices Π^* and Π which are used as the basis of discussion in the first section of the third chapter; the second (POPGEN) generated the theoretical population of response patterns cited in the second section of the same chapter. All programming was done in FORTRAN II language with the exception of a random number generator (XRECTF) which was coded in FAP; this was used to sample from the theoretical population previously mentioned. Perhaps the greatest difficulty encountered was the search

for a routine to accurately determine the latent roots and latent vectors (eigenvalues and eigenvectors) of a real non-symmetric matrix. This search was in progress for approximately one month when an excellent program (MATVEC) was located; this was modified and used as a subroutine in all versions of the LSA program. Credit for the programming of MATVEC is given to L. W. Ehrlich of the University of Texas. The matrix inversion routine (INVERT) in LSA was extracted from BIMD06, one of a series of statistical programs written by the Biomedical Data Processing Group of the University of California at Los Angeles.

This project was made possible by support for computer time from the University of Maryland and the National Aeronautics and Space Administration under grant NsG-398; the facilities of the University's Computer Science Center were used for all computations. The author wishes to express deep appreciation to his thesis advisor, Dr. E. F. Heermann, Associate Professor of Psychology, for guidance and an introduction to both matrix algebra and the latent class model, to Dr. J. M. Ortega, Research Assistant Professor of the Computer Science Center, for assistance in the resolution of several methodological problems, and to Drs. C. J. Bartlett and L. O. Walder, Associate Professors of Psychology, for serving on his thesis committee.

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CHAPTER I

THE CONCEPT OF A LATENT STRUCTURE

Rather than begin with a discussion of what latent structure analysis is and of how it originated, an initial example might prove to be beneficial in understanding the material which is to follow. Suppose we have two groups of subjects who respond to the same pair of test items, each item measuring a single trait and having only two alternatives. Table 1 gives a summary of responses as they might appear.

Table 1

Response Patterns of Two Groups on Two Dichotomous Items

	Group 1				Group 2				Total		
	+	-			+	-			+	-	
+	4	24	28	+	36	2	38	+	40	26	66
-	8	48	56	-	18	1	19	-	26	49	75
	12	72	84		54	3	57		66	75	141

It is readily noticeable that unequal numbers of subjects are in each group and that the patterns of responding are different. Closer inspection reveals that, in both groups, the response to the first item is unrelated to the response to the second; the probability of a positive response to both items is equal to the product of the probabilities of a positive response to each separate item, that is to say, $p_{12} = p_1 p_2$. (The designation of one of the alternatives as positive and the other as negative is an arbitrary matter.) Merging the two groups into one (N = 141), we note that the total group's response pattern does not exhibit the property of statistical independence; in other words, $p_{12} \neq p_1 p_2$. In an actual situation, only the overall pattern of responding to the two dichotomous items might be known or manifest; the response patterns of the subgroups, although they exist, might be obscured or latent. Thus, despite independent responding in each of the two groups in Table 1, "clouding" results when the respondents from different groups are brought together to form the manifest response pattern for both items.

If a population consists of m distinct groups where $m \geqq 2$, a latent structure exists if the responding to dichotomous items is independent in the probabilistic sense within each group and if the characteristics which serve to distingush each group are not known. These groups are formally called latent classes. Setting the definition of ν^{α} as the proportion of respondents in the α-th latent class (where $\sum_{\alpha=1}^{m}\nu^{\alpha} = 1$), λ_i^{α} as the probability that a person from the α-th class responds positively to item i, and π_i as the probability that a person from the total population responds positively to item i, the manifest response pattern of the population can be explained or accounted for by using the parameters for each latent class in equation form as shown below in (1). The highest order joint manifest probability will have K subscripts where K is the number of items. The relationships given in (1), however, do not extend beyond third order joint probabilities for economy of space and for reasons which will become obvious later.

$$\pi_i = \sum_{\alpha=1}^{m}\nu^{\alpha}\lambda_i^{\alpha}$$

$$\pi_{ij} = \sum_{\alpha=1}^{m}\nu^{\alpha}\lambda_i^{\alpha}\lambda_j^{\alpha} \tag{1}$$

$$\pi_{ijk} = \sum_{\alpha=1}^{m}\nu^{\alpha}\lambda_i^{\alpha}\lambda_j^{\alpha}\lambda_k^{\alpha}$$

It becomes readily apparent that statistical independence must prevail within each latent class in order for these relationships to be valid. Briefly stating the purpose of latent structure analysis as it will be covered in this paper, the latent probabilities (νs and λs) associated with each class are to be estimated from the manifest probabilities (πs) on the assumption that a latent structure does indeed exist.

Having set forth this preliminary example, we shall cover the origin and development of latent structure analysis. The work of Lazarsfeld [1950] can be cited as the initial research in this area. The manifest material has, in this original investigation and in those which followed, consisted of qualitative dichotomies. Each element of this material is referred to as an "item"; it may be, for example, a dichotomous item in a questionnaire but the definition also includes a dichotomized observation of overt behavior. An "item list" consists of a series of items ordered in a fixed but arbitrary way, the response to each item being scored either positive (+) or negative (-). One subject's response pattern is part of the manifest material and is composed of a series of scores designated either "+" or "-". From a sample, we may determine the proportion of persons who respond positively to any item, pair of items, triplicate of items, etc. or p_i, p_{ij}, p_{ijk}, etc. respectively. These are estimates of the population values π_i, π_{ij}, π_{ijk}, etc. and will be used to estimate the latent parameters.

An important part of Lazarsfeld's work is the notion of an item trace line. Assuming that a one-dimensional continuum (x) exists and that the probability of a person's responding positively to item i is a function of his position on x, $f_i(x)$ represents the trace line of item i. Theoretically, for respondents with the same value of x, nothing else relates one item to another since the effect of the underlying continuum has been removed; hence, the trace line for a joint positive response is a multiplicative function of the trace lines for each individual item. This is given in equation form in (2).

$$f_{ijk\ldots}(x) = f_i(x)f_j(x)f_k(x)\ldots \qquad (2)$$

The statistical independence expressed here is identical to that which has been set forth previously, that is, for those at one point on the continuum or in one latent class, a joint positive response is the product of the probabilities for each item taken separately. In reality, of course, this condition of independence may not be fulfilled due to random sampling error.

Since respondents are spread along continuum x, the population may be described by a probability density function $\phi(x)$ where the proportion of persons in each small interval dx is equal to $\phi(x)dx$. Given the trace lines for the items i, j, and k, we have

$$\pi_i = \int_{-\infty}^{\infty} f_i(x)\phi(x)\,dx \quad ,$$

$$\pi_{ij} = \int_{-\infty}^{\infty} f_i(x)f_j(x)\phi(x)\,dx \quad , \text{ and} \tag{3a}$$

$$\pi_{ijk} = \int_{-\infty}^{\infty} f_i(x)f_j(x)f_k(x)\phi(x)\,dx \quad .$$

With reference to the equations $y = f(x)$ and $y = \phi(x)$, x represents the same latent continuum but y is used in two different ways. In the former equation, y is the proportion of people at a specific point on the continuum (those between x and x + dx) who respond positively to an item; this usage has been previously encountered in the definition of λ. The latter equation defines y as the proportion of people from the total population who are located at this particular point; this may be equated to the parameter ν.[1] Passing from (3a) to more specific equations involving moments of the function $\phi(x)$ and assuming a linear[2] trace line model for all items to promote simplicity, a revision of the original relationships is given in (3b) where $\int_{-\infty}^{\infty} \phi(x)\,dx = 1$ by definition and $\int_{-\infty}^{\infty} x^k\phi(x)\,dx = M_k$, the k-th moment of the distribution $y = \phi(x)$.

$$\pi_i = \int_{-\infty}^{\infty} [a_i^0 + a_i^1 x]\phi(x)\,dx = a_i^0\int_{-\infty}^{\infty}\phi(x)\,dx + a_i^1\int_{-\infty}^{\infty} x\phi(x)\,dx$$

$$\pi_{ij} = \int_{-\infty}^{\infty} [a_i^0 + a_i^1 x]\,[a_j^0 + a_j^1 x]\phi(x)\,dx$$

$$= a_i^0 a_j^0\int_{-\infty}^{\infty}\phi(x)\,dx + [a_i^0 a_j^1 + a_i^1 a_j^0]\int_{-\infty}^{\infty} x\phi(x)\,dx + a_i^1 a_j^1\int_{-\infty}^{\infty} x^2\phi(x)\,dx$$

$$\tag{3b}$$

$$\pi_{ijk} = \int_{-\infty}^{\infty} [a_i^0 + a_i^1 x]\,[a_j^0 + a_j^1 x]\,[a_k^0 + a_k^1 x]\phi(x)\,dx$$

$$= a_i^0 a_j^0 a_k^0\int_{-\infty}^{\infty}\phi(x)\,dx + [a_i^0 a_j^0 a_k^1 + a_i^0 a_j^1 a_k^0 + a_i^1 a_j^0 a_k^0]\int_{-\infty}^{\infty} x\phi(x)\,dx + \ldots$$

$$[a_i^0 a_j^1 a_k^1 + a_i^1 a_j^0 a_k^1 + a_i^1 a_j^1 a_k^0]\int_{-\infty}^{\infty} x^2\phi(x)\,dx + a_i^1 a_j^1 a_k^1\int_{-\infty}^{\infty} x^3\phi(x)\,dx$$

Finally, we may reduce (3b) to the form shown in (3c) where the weighting factors have been redefined as follows: $a_{ij}^0 = a_i^0 a_j^0$, $a_{ij}^1 = a_i^0 a_j^1 + a_i^1 a_j^0$, $a_{ij}^2 = a_i^1 a_j^1$, $a_{ijk}^0 = a_i^0 a_j^0 a_k^0$, $a_{ijk}^1 = a_i^0 a_j^0 a_k^1 + a_i^0 a_j^1 a_k^0 + a_i^1 a_j^0 a_k^0$, $a_{ijk}^2 = a_i^0 a_j^1 a_k^1 + a_i^1 a_j^0 a_k^1 + a_i^1 a_j^1 a_k^0$, and $a_{ijk}^3 = a_i^1 a_j^1 a_k^1$. After making appropriate substitutions, the equations in (3b) are represented as shown in (3c).

The relationships in (3) indicate the assumed ties between the latent trace lines and population distribution

$$\pi_i = a_i^0 + a_i^1 M_1 \quad ,$$

$$\pi_{ij} = a_{ij}^0 + a_{ij}^1 M_1 + a_{ij}^2 M_2 \quad , \text{ and} \tag{3c}$$

$$\pi_{ijk} = a_{ijk}^0 + a_{ijk}^1 M_1 + a_{ijk}^2 M_2 + a_{ijk}^3 M_3 \quad .$$

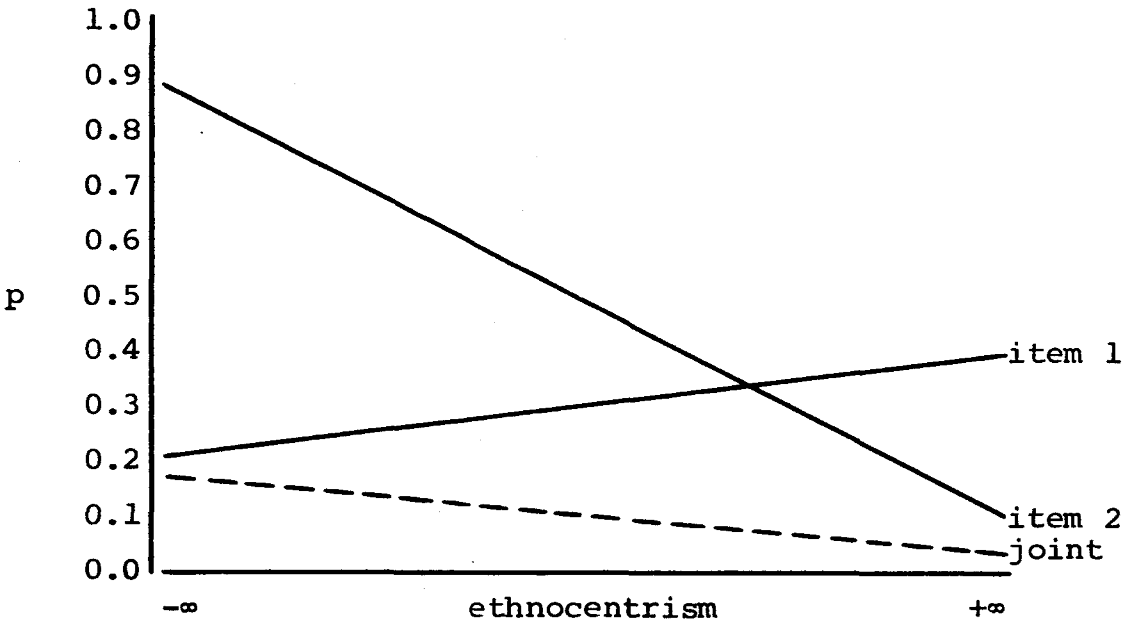
(on the right side of these equations) and the manifest response patterns (on the left side). They are called accounting equations because they explain or account for the manifest parameters in terms of the latent ones. Lazarsfeld provided a solution for the trace line parameters from the manifest data. Since the actual computations posed do not aid in the development of this paper, they will not be presented.

With reference to an assumed one-dimensional continuum of ethnocentrism as an example, it might be postulated that the positions of people distributed along this continuum determined their probabilities of choosing the positively designated alternatives of dichotomous items. Ideally, the criterion for selecting items for inclusion in the item list should be that they discriminate between various levels of this attitude, that is, that each item have a differential relationship to the underlying continuum. Using a linear model, two hypothetical trace lines and the joint response

Figure 1

Hypothetical Trace Lines of Two Items in a Test

of Ethnocentrism



trace line (dashed line) are shown in Figure 1. It is obvious that as a respondent's degree of ethnocentrism becomes greater, the probability of a positive response to item 1 increases while that to item 2 decreases. As was pointed out earlier, the critical assumption of the latent structure model is that responding to the items is independent for each small interval between x and dx on the continuum, that is, that joint relationships among items are completely explained by the relationship of each item to the latent continuum. As is shown by (2), the probability associated with each point on the joint response trace line is equal to the product of the probabilities associated with the corresponding points on the item trace lines; for any specific value of x, $p_{12} = p_1 p_2$. Restating what has been previously established, the purpose of latent structure analysis as derived by Lazarsfeld is to estimate the latent parameters of the item trace lines.

A distinction should be made between the latent structure model as given in the preceding summary of Lazarsfeld's work and the latent class model which is an extension of it. An assumption of the latent structure model is local independence, that is, that persons at the same position on the underlying continuum x respond independently to different items. The latent class model also assumes local independence but, in addition, postulates that the population of respondents can be represented by a finite number of points or classes on the continuum or, in other words, that $\phi(x)$ is discrete. Therefore, every person in the population can be placed in one of several distinct groups which vary as to their position on x. This may be considered as a simplifying condition and the latent class model can be viewed as a special case of the latent structure model. It may now be pointed out that the example given at the beginning of this chapter was concerned with the latent class model and that the relationships in (1) are finite accounting equations for m groups or points on the continuum x.

It is apparent that the latent class model could have been developed without utilizing the concepts of latent continuum and item trace line.[3] All that is necessary is the assumption that m subgroups or latent classes exist and that they possess the properties of intra-class statistical independence and inter-class uniqueness. Thus, the classes need not necessarily be ordered along a continuum and, as a result, the latent class model becomes even more general with respect to the situations to which it might be applicable. For the remainder of this paper, we will be occupied with the latent class model conceived in this manner. Although latent structure analysis originated as a technique in attitude measurement, it is interesting to note that its potential usefulness in other areas of psychological research was initially recognized by Lazarsfeld [1950, p. 365]. "No limitation is set on the kind of dichotomies which can be used. It would not change the theory, for instance, if some of the items were observations to the effect that each person did or did not perform a certain act or own a certain object." We should therefore keep in mind that the material to follow is easily generalized to any area where responses can be reduced to binary patterns.

CHAPTER II

THE LATENT CLASS MODEL

Stemming from the work of Lazarsfeld, solutions for the latent class model have been derived by Green [1951] and Anderson [1954]. As these are closely related to the present research, it is necessary that they be covered in detail in this chapter.

Green proposed a solution which was dependent upon manifest probabilities of the first, second, and third orders only.[4] For each of m latent classes, the proportion of the population in class s is equal to n_s ($\sum_{s=1}^{m} n_s = 1$) and the probability that a person in class s will respond positively to item i equals v_{is} where s = 1, 2, ..., m and, r being the total number of items, i = 0, 1, ..., r. The accounting equations of Green's procedure are

$$p_i = \sum_{s=1}^{m} n_s v_{is} \quad ,$$

$$p_{ij} = \sum_{s=1}^{m} n_s v_{is} v_{js} \quad , \text{ and} \tag{4}$$

$$p_{ijk} = \sum_{s=1}^{m} n_s v_{is} v_{js} v_{ks} \quad .$$

It should be noted that the relationships in (4) are identical in meaning to those in (1).

The following matrices are used by Green in the development of his solution. P_0 is given as a symmetric matrix of order r+1 which consists of manifest proportions. The elements in the first row and the first column are the first order probabilities and are referred to as "manifest marginals" or, more simply, as "marginals" since they border this matrix. The remainder of P_0 is made up of the second order probabilities for all possible combinations of items. P_k is a matrix of similar structure in which the k-th item is used as a "stratifier"; the choice of this item determines which third order probabilities become elements of P_k. The manifest matrices P_0 and P_k are defined in (5) and (6). As to

$$P_0 = \begin{bmatrix} 1 & p_1 & p_2 & \cdots & p_r \\ p_1 & p_{11} & p_{12} & \cdots & p_{1r} \\ p_2 & p_{21} & p_{22} & \cdots & p_{2r} \\ \vdots & \vdots & \vdots & & \vdots \\ p_r & p_{r1} & p_{r2} & \cdots & p_{rr} \end{bmatrix} \tag{5}$$

the structure of these matrices, double subscripted elements in P_0 are abbreviated such that, for example, $p_{00} = 1$ and $p_{i0} = p_i$; triple subscripted elements in P_k are shortened in

$$P_k = \begin{bmatrix} p_k & p_{1k} & p_{2k} & \cdots & p_{kk} & \cdots & p_{rk} \\ p_{1k} & p_{11k} & p_{12k} & \cdots & p_{1kk} & \cdots & p_{1rk} \\ p_{2k} & p_{21k} & p_{22k} & \cdots & p_{2kk} & \cdots & p_{2rk} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ p_{kk} & p_{k1k} & p_{k2k} & \cdots & p_{kkk} & \cdots & p_{krk} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ p_{rk} & p_{r1k} & p_{r2k} & \cdots & p_{rkk} & \cdots & p_{rrk} \end{bmatrix} \tag{6}$$

a similar way, that is, $p_{00k} = p_k$ and $p_{0jk} = p_{jk}$. Thus, subscripts are suppressed for clarity when i and/or j is equal to zero.

It should be pointed out that, although P_0 and P_k are

$$p_{ii} = \sum_{s=1}^{m} n_s v_{is}^2 \quad ,$$

$$p_{iij} = \sum_{s=1}^{m} n_s v_{is}^2 v_{js} \quad , \tag{7}$$

$$p_{kkk} = \sum_{s=1}^{m} n_s v_{ks}^3 \quad , \text{ etc.}$$

considered to be manifest data matrices, they do contain elements which cannot be directly computed from observed response patterns. Due to the way in which these matrices were formed, some entries will have recurring subscripts as shown by (7). These parameters, as we will see, become a major problem in Green's solution of the latent class model.

In addition to the matrices of manifest parameters, the latent parameters are also given in matrix form. The proportion of the population in each latent class becomes a diagonal entry of the matrix N which is necessarily of order m and is represented as

$$N = \begin{bmatrix} n_1 & 0 & \cdots & 0 \\ 0 & n_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & n_m \end{bmatrix} . \tag{8}$$

The item probabilities for each latent class are elements of the (r+1) x m matrix L as given by (9). The first row of L is used to represent a dummy item, that is, $v_{0s} = 1$; its inclusion in L is a prerequisite for the derivation of Green's solution. The latent parameters associated with a single item or, in other words, the entries in a single row of the

$$L = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ v_{11} & v_{12} & \cdots & v_{1m} \\ v_{21} & v_{22} & \cdots & v_{2m} \\ \vdots & \vdots & & \vdots \\ v_{k1} & v_{k2} & \cdots & v_{km} \\ \vdots & \vdots & & \vdots \\ v_{r1} & v_{r2} & \cdots & v_{rm} \end{bmatrix} \tag{9}$$

matrix L, are also specified as the elements of a diagonal matrix of order m. Thus, for item k, we have

$$D_k = \begin{bmatrix} v_{k1} & 0 & \cdots & 0 \\ 0 & v_{k2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & v_{km} \end{bmatrix} \tag{10}$$

Using P_0, P_k, N, L, and D_k and assuming that the conditions in (4) are fulfilled, it is possible to establish accounting equations in matrix form which relate the manifest and latent matrices. These are expressed by (11) and (12).

$$P_0 = LNL^T \tag{11}$$

$$P_k = LND_kL^T \tag{12}$$

Since the use of D_k results in third order probabilities stratified only by item k, all probabilities of the third order can be accounted for by considering r-1 similar matrices, one for each of the remaining items. Substituting each of these in (12), we have r-1 additional P_k matrices. This leads us to the definition of $D_{(1)}$ and $P_{(1)}$ as follows:

$$D_{(1)} = \sum_{k=1}^{r} D_k \quad \text{and} \tag{13}$$

$$P_{(1)} = \sum_{k=1}^{r} P_k \quad . \tag{14}$$

Accounting for all possible third order probabilities simultaneously, we may replace (12) by

$$P_{(1)} = LND_{(1)}L^T \quad . \tag{15}$$

Having established the accounting equations of Green's solution in (11) and (15), we may relate the present model to that of Lazarsfeld. The latter was concerned with the calculation of trace line parameters, that is, the probability of a positive response to items for persons at different positions on a latent continuum. The goal of the former is to determine the elements of N and L or, more specifically, to estimate the proportion of the population in each latent class and, for those in each class, the probability of responding positively to each item. The difference between the two models is the assumption of a point versus a continuous distribution of respondents.

Because N is a diagonal matrix, (11) can be written as shown in (16) where $N^{\frac{1}{2}} = [N^{\frac{1}{2}}]^T$.

$$P_0 = LN^{\frac{1}{2}}N^{\frac{1}{2}}L^T = [LN^{\frac{1}{2}}][LN^{\frac{1}{2}}]^T \tag{16}$$

Factoring P_0 results in a new matrix as given by

$$P_0 = BB^T \quad . \tag{17}$$

The matrix B differs from $LN^{\frac{1}{2}}$ in (16) by an orthogonal transformation such that

$$LN^{\frac{1}{2}} = B\Lambda_b \tag{18}$$

where $\Lambda_b^T \Lambda_b = I$. With reference to (15), $P_{(1)}$ may be revised as follows:

$$P_{(1)} = [LN^{\frac{1}{2}}] D_{(1)} [LN^{\frac{1}{2}}]^T \quad . \tag{19}$$

Combining (18) and (19),

$$P_{(1)} = [B\Lambda_b] D_{(1)} [B\Lambda_b]^T = B\Lambda_b D_{(1)} \Lambda_b^T B^T \quad . \tag{20}$$

Defining the matrix Q, we have

$$Q = \Lambda_b D_{(1)} \Lambda_b^T \tag{21}$$

and therefore,

$$P_{(1)} = BQB^T \quad . \tag{22}$$

Premultiplying by B^T and postmultiplying by B, (22) is modified to the form

$$B^T P_{(1)} B = B^T BQB^T B \tag{23}$$

where a solution for Q gives

$$Q = [B^T B]^{-1} B^T P_{(1)} B [B^T B]^{-1} \quad . \tag{24}$$

From (24)[5], we can determine Q and, using this matrix, $D_{(1)}$ and Λ_b. $D_{(1)}$ is a diagonal matrix of the latent roots of Q and Λ_b, a matrix of the latent vectors; these three matrices are related as shown in (21). With B and Λ_b known, N and L can be determined. By referring to (18), the entries in the first row of $B\Lambda_b$ are equal to the square roots of the diagonal elements in N since the first row of L represents a dummy item, all entries being equal to unity. After N has been found, L is readily available. The main restrictions on this solution are that L be of rank m and that each diagonal element of $D_{(1)}$ be non-zero and different from all others.

In order to eliminate the excessive computations inherent in this method as well as the troublesome estimation of manifest probabilities with recurring subscripts, Anderson modified Green's procedure to what may be called an asymmetric approach since all the manifest data is not utilized. Reference is made to the situation in which K dichotomous items are responded to.[6] The proportions of people in the population who respond positively to item i, items i and j, and items i, j, and k are denoted by π_i, π_{ij}, and π_{ijk} in that order. For each of m latent classes, the proportion of respondents in class α is equal to ν^{α} where $\alpha = 1, 2, \ldots, m$ and the probability that a person in class α will respond positively to item i is equal to λ_i^{α}. The manifest πs are functions of the latent νs and λs and are related as shown in (1). The similarity between the two solutions becomes evident with the recognition that the accounting equations of Anderson and Green (given by (1) and (4) respectively) specify identical relationships. The difference arises from the restriction imposed by Anderson which is that all manifest parameters to be used in his solution have nonrecurring subscripts. The derivation of Anderson's procedure follows.

Defining the m x (K+1) matrix Λ, we have

$$\Lambda = \begin{bmatrix} 1 & \lambda_1^1 & \lambda_2^1 & \cdots & \lambda_K^1 \\ 1 & \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_K^2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_1^m & \lambda_2^m & \cdots & \lambda_K^m \end{bmatrix} \tag{25}$$

where, from the previous definitions, classes are referred to by superscripts and items, by subscripts. Following what was done by Green, a dummy item is defined such that $\lambda_0^\alpha = 1$. Λ_1 is given as an m x m matrix formed by the first m columns of Λ; Λ_2 has the same dimensions and consists of the first column and the next m-1 columns of Λ not utilized in the formation of Λ_1. These matrices are represented as follows:

$$\Lambda_1 = \begin{bmatrix} 1 & \lambda_1^1 & \lambda_2^1 & \cdots & \lambda_{m-1}^1 \\ 1 & \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{m-1}^2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_1^m & \lambda_2^m & \cdots & \lambda_{m-1}^m \end{bmatrix} \quad \text{and} \tag{26}$$

$$\Lambda_2 = \begin{bmatrix} 1 & \lambda_m^1 & \lambda_{m+1}^1 & \cdots & \lambda_{2m-2}^1 \\ 1 & \lambda_m^2 & \lambda_{m+1}^2 & \cdots & \lambda_{2m-2}^2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_m^m & \lambda_{m+1}^m & \cdots & \lambda_{2m-2}^m \end{bmatrix} . \tag{27}$$

Choosing an item not used in the construction of either Λ_1 or Λ_2, we have the m x m diagonal matrix Δ as shown by (28). It should be noted that, from the total set of K items, two subsets of m-1 items plus one additional item have been

$$\Delta = \begin{bmatrix} \lambda_k^1 & 0 & \cdots & 0 \\ 0 & \lambda_k^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_k^m \end{bmatrix} \tag{28}$$

selected, the extra item serving as a stratifier. Thus, K must be equal to or greater than 2m-1. Of course, there is no way to check this since the number of latent classes is not known. Specifying the last matrix of latent content, we have

$$N = \begin{bmatrix} \nu^1 & 0 & \cdots & 0 \\ 0 & \nu^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \nu^m \end{bmatrix} . \tag{29}$$

Each diagonal element of N represents the probability that a respondent drawn randomly from the population is from the corresponding latent class; N is therefore of order m.

In solving for the latent values, we utilize the manifest parameters π_i, π_{ij}, and π_{ijk} where $i = 0, 1, \ldots, m-1$,

$j = 0, m, m+1, \ldots, 2m-2$, and k is a fixed subscript for the stratifying item. The $m \times m$ matrices of manifest content are defined in (30) and (31). Subscripts equal to zero have been

$$\Pi^* = \begin{bmatrix} 1 & \pi_m & \pi_{m+1} & \cdots & \pi_{2m-2} \\ \pi_1 & \pi_{1m} & \pi_{1,m+1} & \cdots & \pi_{1,2m-2} \\ \pi_2 & \pi_{2m} & \pi_{2,m+1} & \cdots & \pi_{2,2m-2} \\ \vdots & \vdots & \vdots & & \vdots \\ \pi_{m-1} & \pi_{m-1,m} & \pi_{m-1,m+1} & \cdots & \pi_{m-1,2m-2} \end{bmatrix} \tag{30}$$

$$\Pi = \begin{bmatrix} \pi_k & \pi_{mk} & \pi_{m+1,k} & \cdots & \pi_{2m-2,k} \\ \pi_{1k} & \pi_{1mk} & \pi_{1,m+1,k} & \cdots & \pi_{1,2m-2,k} \\ \pi_{2k} & \pi_{2mk} & \pi_{2,m+1,k} & \cdots & \pi_{2,2m-2,k} \\ \vdots & \vdots & \vdots & & \vdots \\ \pi_{m-1,k} & \pi_{m-1,m,k} & \pi_{m-1,m+1,k} & \cdots & \pi_{m-1,2m-2,k} \end{bmatrix} \tag{31}$$

suppressed. Due to the way in which Π^* and Π were formed, the problem of elements with recurring subscripts has been eliminated. Using the matrices of manifest and latent content, the accounting equations in the solution given by Anderson are presented in (32) and (33). To promote clarity,

$$\Pi^* = \Lambda_1^T N \Lambda_2 \tag{32}$$

$$\Pi = \Lambda_1^T N \Delta \Lambda_2 \tag{33}$$

$$\Lambda_1^T N \Lambda_2 = \tag{34}$$

$$\begin{bmatrix} \Sigma\nu^\alpha & \Sigma\nu^\alpha\lambda_m^\alpha & \Sigma\nu^\alpha\lambda_{m+1}^\alpha & \cdots & \Sigma\nu^\alpha\lambda_{2m-2}^\alpha \\ \Sigma\nu^\alpha\lambda_1^\alpha & \Sigma\nu^\alpha\lambda_1^\alpha\lambda_m^\alpha & \Sigma\nu^\alpha\lambda_1^\alpha\lambda_{m+1}^\alpha & \cdots & \Sigma\nu^\alpha\lambda_1^\alpha\lambda_{2m-2}^\alpha \\ \Sigma\nu^\alpha\lambda_2^\alpha & \Sigma\nu^\alpha\lambda_2^\alpha\lambda_m^\alpha & \Sigma\nu^\alpha\lambda_2^\alpha\lambda_{m+1}^\alpha & \cdots & \Sigma\nu^\alpha\lambda_2^\alpha\lambda_{2m-2}^\alpha \\ \vdots & \vdots & \vdots & & \vdots \\ \Sigma\nu^\alpha\lambda_{m-1}^\alpha & \Sigma\nu^\alpha\lambda_{m-1}^\alpha\lambda_m^\alpha & \Sigma\nu^\alpha\lambda_{m-1}^\alpha\lambda_{m+1}^\alpha & \cdots & \Sigma\nu^\alpha\lambda_{m-1}^\alpha\lambda_{2m-2}^\alpha \end{bmatrix}$$

$$\Lambda_1^T N \Delta \Lambda_2 = \tag{35}$$

$$\begin{bmatrix} \Sigma\nu^\alpha\lambda_k^\alpha & \Sigma\nu^\alpha\lambda_m^\alpha\lambda_k^\alpha & \Sigma\nu^\alpha\lambda_{m+1}^\alpha\lambda_k^\alpha & \cdots & \Sigma\nu^\alpha\lambda_{2m-2}^\alpha\lambda_k^\alpha \\ \Sigma\nu^\alpha\lambda_1^\alpha\lambda_k^\alpha & \Sigma\nu^\alpha\lambda_1^\alpha\lambda_m^\alpha\lambda_k^\alpha & \Sigma\nu^\alpha\lambda_1^\alpha\lambda_{m+1}^\alpha\lambda_k^\alpha & \cdots & \Sigma\nu^\alpha\lambda_1^\alpha\lambda_{2m-2}^\alpha\lambda_k^\alpha \\ \Sigma\nu^\alpha\lambda_2^\alpha\lambda_k^\alpha & \Sigma\nu^\alpha\lambda_2^\alpha\lambda_m^\alpha\lambda_k^\alpha & \Sigma\nu^\alpha\lambda_2^\alpha\lambda_{m+1}^\alpha\lambda_k^\alpha & \cdots & \Sigma\nu^\alpha\lambda_2^\alpha\lambda_{2m-2}^\alpha\lambda_k^\alpha \\ \vdots & \vdots & \vdots & & \vdots \\ \Sigma\nu^\alpha\lambda_{m-1}^\alpha\lambda_k^\alpha & \Sigma\nu^\alpha\lambda_{m-1}^\alpha\lambda_m^\alpha\lambda_k^\alpha & \Sigma\nu^\alpha\lambda_{m-1}^\alpha\lambda_{m+1}^\alpha\lambda_k^\alpha & \cdots & \Sigma\nu^\alpha\lambda_{m-1}^\alpha\lambda_{2m-2}^\alpha\lambda_k^\alpha \end{bmatrix}$$

$\Lambda_1^T N \Lambda_2$ and $\Lambda_1^T N \Delta \Lambda_2$ have been generated in (34) and (35).[7] By inspecting the elements in these product matrices, it is apparent that the condition of statistical independence postulated in (1) is basic to Anderson's solution. The derivation of Green could similarly have been shown to be dependent on the relationships in (4).

The initial step of Anderson's procedure is to find the latent roots of the determinantal equation

$$|\Pi - \theta^{\alpha}\Pi^*| = 0 \quad . \tag{36}$$

Imposing the restriction that Λ_1, Λ_2, and N be nonsingular, the diagonal entries of Δ (symbolically, $\lambda_k^1, \lambda_k^2, \ldots, \lambda_k^m$) can be equated to the roots of (36) by noting that

$$0 = |\Lambda_1^T N \Delta \Lambda_2 - \theta^{\alpha}\Lambda_1^T N \Lambda_2| = |\Lambda_1^T| \cdot |N| \cdot |\Delta - \theta^{\alpha} I| \cdot |\Lambda_2| \quad . \tag{37}$$

With a knowledge of the latent roots (Δ) and the matrices of manifest parameters (Π^* and Π), we shall proceed to determine the remaining latent parameter matrices, that is, Λ_1, Λ_2, and N.

From (36), $[\Pi - \theta^{\alpha}\Pi^*]$ is singular. A latent (column) vector x^{α} exists (excluding the null vector) such that the relations in (38) are true. If the roots are different, each

$$[\Pi - \theta^{\alpha}\Pi^*]x^{\alpha} = 0 \qquad \text{or} \qquad \Pi x^{\alpha} = \theta^{\alpha}\Pi^* x^{\alpha} \tag{38}$$

vector is unique. The elements in a specific vector may be altered by multiplying them by a scalar but any such multiplicative transform of x^{α} will not affect (38). Establishing the roots as entries in the diagonal matrix Θ and the vectors as columns in the matrix X, we have in (39) a summary of the α equations represented by (38). It is required that

$$\Pi X = \Pi^* X\Theta \tag{39}$$

the latent roots of (36) be different from each other to avoid difficulties inherent in the use of (39).

Assuming that $\Theta = \Delta$ by an identical ordering of the roots, a possible solution for X in (39) is Λ_2^{-1}. This is shown, after making appropriate substitutions, by

$$\Lambda_1^T N \Delta \Lambda_2 \Lambda_2^{-1} = \Lambda_1^T N \Lambda_2 \Lambda_2^{-1} \Delta \quad . \tag{40}$$

Rather than state without exception that $X = \Lambda_2^{-1}$, we must take into account the near certainty that the latent vectors in X will be equal to transforms of the columns in Λ_2^{-1}. Defining E_x as a diagonal matrix, each element being the reciprocal of the scalar which transformed the corresponding vector in X, the relationship between X and Λ_2^{-1} is given as

$$X = \Lambda_2^{-1} E_x \quad . \tag{41}$$

Solving (41) for Λ_2, we have

$$\Lambda_2 = E_x X^{-1} \quad . \tag{42}$$

Since the elements in the first column of Λ_2 must all be equal to unity as specified in (27), each diagonal entry in E_x must be the reciprocal of the corresponding element in the first column of X^{-1}.

From (38), transposition of the matrices of manifest parameters gives

$$[\Pi - \theta^{\alpha}\Pi^*]^T y^{\alpha} = 0 \qquad \text{or} \qquad \Pi^T y^{\alpha} = \theta^{\alpha}\Pi^{*T} y^{\alpha} \tag{43}$$

where the equations corresponding to (32) and (33) are

$$\Pi^{*T} = \Lambda_2^T N \Lambda_1 \quad \text{and} \tag{44}$$

$$\Pi^T = \Lambda_2^T N \Delta \Lambda_1 \quad . \tag{45}$$

Referring to (36), latent roots and vectors of the matrix

$\Pi^{*-1}\Pi$ were previously determined; now, after transposition, this matrix is revised to the form $[\Pi^{*T}]^{-1}\Pi^T$. As can be inferred from the notation in (43), the roots of this new matrix are identical to those of the old but the vector associated with each root changes. The former is true since the characteristic equations of these matrices are exactly the same; the latter is true because there has been a change in matrix structure. Following the same line of reasoning as was used before, (46) and (47) equate Y with Λ_1^{-1}.

$$\Pi^T Y = \Pi^{*T} Y \Theta \tag{46}$$

$$\Lambda_2^T N \Delta \Lambda_1 \Lambda_1^{-1} = \Lambda_2^T N \Lambda_1 \Lambda_1^{-1} \Delta \tag{47}$$

The general solution for Y is given by

$$Y = \Lambda_1^{-1} E_Y \tag{48}$$

and, solving for Λ_1, we have

$$\Lambda_1 = E_Y Y^{-1} \quad . \tag{49}$$

The final steps in the solution for the latent matrices Λ_1, Λ_2, and N are accomplished in the following manner. With reference to (32) and (41), we can establish that

$$\Pi^{*}X = [\Lambda_1^T N \Lambda_2][\Lambda_2^{-1} E_x] = \Lambda_1^T N E_x \tag{50}$$

and similarly, from (32) and (48),

$$\Pi^{*T}Y = [\Lambda_2^T N \Lambda_1][\Lambda_1^{-1} E_y] = \Lambda_2^T N E_y \quad . \tag{51}$$

Hence,

$$\Lambda_1^T = \Pi^{*}X[NE_x]^{-1} \quad \text{and} \tag{52}$$

$$\Lambda_2^T = \Pi^{*T}Y[NE_y]^{-1} \quad . \tag{53}$$

Since the first row of both Λ_1^T and Λ_2^T consists entirely of 1s, the diagonal elements of NE_x and NE_y are equal to the entries in the first row of $\Pi^{*}X$ and $\Pi^{*T}Y$ respectively. It should be brought to mind that NE_x and NE_y are diagonal matrices and that the inverse of each is a diagonal matrix in which the elements are simply the reciprocals of those in the original matrix. Thus, it could have also been stated that the diagonal entries of NE_x^{-1} and NE_y^{-1} are the reciprocals of the entries in the first row of $\Pi^{*}X$ and $\Pi^{*T}Y$ in that order. Having found Λ_1^T and Λ_2^T in (52) and (53), Λ_1 and Λ_2

are easily determined, NE_x^{-1} and NE_y^{-1} being by-products of computation.

N is now the only latent matrix to be defined in terms of known matrices. Referring to (41) and (48), we note that

$$\Lambda_2 X = \Lambda_2 \Lambda_2^{-1} E_x = E_x = N^{-1}[NE_x] \quad \text{and} \tag{54}$$

$$\Lambda_1 Y = \Lambda_1 \Lambda_1^{-1} E_y = E_y = N^{-1}[NE_y] \quad . \tag{55}$$

Having already found NE_x^{-1} and NE_y^{-1}, either of these may be used to compute N^{-1} since, from (54) and (55),

$$N^{-1} = \Lambda_2 X[NE_x]^{-1} \quad \text{and} \tag{56}$$

$$N^{-1} = \Lambda_1 Y[NE_y]^{-1} \quad . \tag{57}$$

The reciprocals of the diagonal elements of N^{-1} are the diagonal elements of N.

At this point, we have covered the method given by Anderson for the derivation of the matrices of latent content; the data consisted of manifest parameters for which the conditions specified in (1) are perfectly fulfilled. In practice, however, we can only estimate the true manifest values. For example, the manifest data matrices Π^* and Π are error-free, that is, they are reproduced perfectly by $\Lambda_1^T N \Lambda_2$ and $\Lambda_1^T N \Delta \Lambda_2$ respectively. On sampling from the population, the data matrices P* and P will not be completely accounted for by the product matrices mentioned above since the sample will be in error to some degree. This does not invalidate the latent class model because any method of estimation is used and qualified in light of random fluctuations in the data. However, we should realize that $P^* \neq \Pi^*$ and $P \neq \Pi$. The relationships associated with any application of Anderson's method (assuming that latent classes do exist) are similar to (32) and (33) as shown by

$$P^* = \Lambda_1^T N \Lambda_2 + E_1 = \Pi^* + E_1 \quad \text{and} \tag{58}$$

$$P = \Lambda_1^T N \Delta \Lambda_2 + E_2 = \Pi + E_2 \quad ; \tag{59}$$

E_1 and E_2 are m x m error matrices. Since this solution will only approximate the latent matrices from contaminated manifest matrices, equations of a more general form are given in (60) and (61) where L_1, L_2, $\hat{N}$, and D are estimates of Λ_1, Λ_2,

$$P^* = L_1^T \hat{N} L_2 \tag{60}$$

$$P = L_1^T \hat{N} D L_2 \tag{61}$$

N, and Δ in that order. More will be said about (60) and (61) in the next chapter.

CHAPTER III

THE SENSITIVITY OF THE METHOD

To the present time, the latent class model of latent structure analysis has never undergone an empirical test. The work of Green followed by that of Anderson prompted several writers as Gibson [1955] and McHugh [1956] to suggest ways of improving the method but, again, demonstrations concerning practical applications of the original procedure and/or its revisions are nonexistent. In this chapter, we will cover an investigation of the sensitivity of latent class analysis to error in the manifest data matrices. The specific method to be tested will be that of Anderson since, in the opinion of the author, the effect of random error on estimates of the latent parameters will be less with this procedure than with that of Green.

It is assumed that the absence of empirical work with latent class analysis is due to the excessive and complex computations inherent in all variations of this technique. This barrier can be minimized by the use of an electronic digital computer. A computer program (in FORTRAN II language) has been written to carry out the computations given by Anderson in his solution.[8] Thus, we have the means to make this investigation feasible. In line with computer utilization, it should be noted that Anderson's derivation of the latent parameters involves matrix manipulations which were formulated primarily to facilitate hand calculation. Since this barrier has been eliminated, several modifications have been made in his procedure. This revised method will now be presented; statistics rather than parameters are used in these equations.

The first major computation by the program concerns a solution for the latent roots, that is, the diagonal entries of D. From (36), we have the equivalent form

$$|P^{*-1}P - t^{\alpha}I| = 0 \quad . \tag{62}$$

The matrix $P^{*-1}P$ is computed and its roots are determined. Similarly, with reference to (38), corresponding forms are given by (63). The latent vector $\hat{x}^{\alpha}$ is found after the solu-

$$[P^{*-1}P - t^{\alpha}I]\hat{x}^{\alpha} = 0 \qquad \text{or} \qquad [P^{*-1}P]\hat{x}^{\alpha} = t^{\alpha}\hat{x}^{\alpha} \tag{63}$$

tion for the root t^{α}.

Because $\hat{X}$ and $\hat{Y}$ are derived from the same manifest matrices, they are related to one another. The solution for $\hat{Y}$ from $\hat{X}$ is expressed in (64). The validity of this can be

$$\hat{Y} = [\hat{X}^{-1}P*^{-1}]^T = [P*^{-1}]^T[\hat{X}^T]^{-1} \tag{64}$$

confirmed by substituting (64) in (46); the result is shown in (65). Observe that (65c) is of the same form as (39). The

$$P^T[P*^{-1}]^T[\hat{X}^T]^{-1} = P*^T[P*^{-1}]^T[\hat{X}^T]^{-1}D \tag{65a}$$

$$\hat{X}^{-1}P*^{-1}P = D\hat{X}^{-1}P*^{-1}P* \tag{65b}$$

$$P\hat{X} = P*\hat{X}D \tag{65c}$$

latent matrices L_2 and L_1 are related to $\hat{X}$ and $\hat{Y}$ as shown by (42) and (49) respectively. By dividing the elements in each row of $\hat{X}^{-1}$ by the first element in that row, the entries in the first column of the resulting L_2 matrix are all equal to unity as defined. Using $\hat{Y}^{-1}$ in the same manner, L_1 can be determined. From (64), we can derive $\hat{Y}^{-1}$ directly as has been done in (66) and therefore, $\hat{Y}$ need not be computed. The

$$\hat{Y} = [P*^T]^{-1}[\hat{X}^T]^{-1} \tag{66a}$$

$$\hat{Y}^{-1} = \hat{X}^TP*^T \tag{66b}$$

$$\hat{Y}^{-1} = [P*\hat{X}]^T \tag{66c}$$

$$\hat{N} = [L_1^T]^{-1}P^*L_2^{-1} \tag{67}$$

solution for $\hat{N}$ is derived from (32) and given by (67). At this point, the revision of Anderson's method for computing the matrices L_1, L_2, D, and $\hat{N}$ has been presented.

1. Introduction of Error into Π^* and Π

The discussion will now be centered on a description of

$$\Lambda_1 = \begin{bmatrix} 1 & 0.9 & 0.2 \\ 1 & 0.7 & 0.9 \\ 1 & 0.1 & 0.1 \end{bmatrix}$$

$$\Lambda_2 = \begin{bmatrix} 1 & 0.8 & 0.4 \\ 1 & 0.4 & 0.8 \\ 1 & 0.3 & 0.3 \end{bmatrix}$$

$$(68)$$

$$N = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

the procedure by which the latent class model was tested for sensitivity to error in the manifest data matrices. This could not have been accomplished on the theoretical level; rather, the only approach to the problem was an empirical one. Utilizing (32) and (33), we can generate the manifest matrices from latent givens. In the present investigation, the number of latent classes is set at three or, in other words, $m = 3$.[9]

The initial step taken was the specification of Λ_1, Λ_2, N, and Δ. All of these matrices were established by using the theoretical structure given by Anderson in his article; they are defined in (68). The product matrices Π^* and Π are shown in (69).[10] Note that, in (68) and (69), elements are

$$\Pi^* = \begin{bmatrix} 1.00000 & 0.50000 & 0.58000 \\ 0.64000 & 0.36200 & 0.39400 \\ 0.53000 & 0.23400 & 0.39000 \end{bmatrix}$$

(69)

$$\Pi = \begin{bmatrix} 0.54000 & 0.32200 & 0.31400 \\ 0.42000 & 0.26500 & 0.23780 \\ 0.28100 & 0.13380 & 0.20220 \end{bmatrix}$$

specified as population values.

The computer program written to perform the operations given by Anderson may now be applied. To restate the purpose of these computations, we wish to obtain estimates of Λ_1, Λ_2, N, and D (namely L_1, L_2, $\hat{N}$, and D) from P* and P which are estimates of Π^* and Π respectively. By beginning with error-free P* and P matrices, that is, matrices which are identical to Π^* and Π as given in (69), we can state that discrepancies between the estimated matrices and those in (68) result from the operations performed. With this in mind,

$$L_1 = \begin{bmatrix} 1.00000 & 0.10000 & 0.10000 \\ 1.00000 & 0.70000 & 0.90000 \\ 1.00000 & 0.90000 & 0.20001 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1.00000 & 0.30000 & 0.30000 \\ 1.00000 & 0.40000 & 0.80000 \\ 1.00000 & 0.80000 & 0.40000 \end{bmatrix}$$

$$\tag{70}$$

$$\hat{N} = \begin{bmatrix} 0.20000 & -0.00000 & -0.00000 \\ -0.00000 & 0.50000 & -0.00000 \\ 0.00000 & -0.00000 & 0.30000 \end{bmatrix}$$

$$D = \begin{bmatrix} 0.10000 & 0 & 0 \\ 0 & 0.50000 & 0 \\ 0 & 0 & 0.90000 \end{bmatrix}$$

reference is made to the computed matrices in (70). Two observations should be made; the first is that the first and third rows of the matrices in (68) have been interchanged in (70). This can be explained by referring to (40) where it was assumed that $\Theta = \Delta$. While this supposition was involved in the derivation of the method, it need not be fulfilled in the utilization of it. In the present example, the order in which the latent roots were computed (d_{11} first, d_{22} second, etc.) does not correspond to the order in which they were set up in Δ. Expressing this differently, while $d_{22} = \delta_{22}$, $d_{11} = \delta_{33}$ and $d_{33} = \delta_{11}$. The second observation to be made concerns the fact that the theoretical and computed matrices differ only in one element, $L_{1(3,3)}$. It could be shown that the displacement of rows does not affect computational accuracy. Thus, the small discrepancy noted above is not due to a row interchange but, instead, might be attributed to the complexity of computations involved as, for example, in latent root determination. We must conclude that computations are performed with sufficient accuracy to provide good estimates of the latent parameters from error-free manifest matrices.[11]

For checking the accuracy of computations, the product of the estimated latent matrices is given in (71). This matrix is identical to Π as shown in (69) and is evidence

$$L_1^T\hat{N}DL_2 = \begin{bmatrix} 0.54000 & 0.32200 & 0.31400 \\ 0.42000 & 0.26500 & 0.23780 \\ 0.28100 & 0.13380 & 0.20220 \end{bmatrix} \tag{71}$$

that, although the estimates are slightly discrepant, computations are sufficiently accurate to make possible reproduction of the original data matrix Π. The matrix $L_1^T\hat{N}L_2$ was not chosen for checking purposes since P* was used to estimate N as shown by (67).

In order to test the sensitivity of the latent class model, error will be introduced into the manifest matrices in accordance with the relations given in (58) and (59). Randomly varying the elements of Π^* and Π in the fifth decimal place, we have

$$P^* = \begin{bmatrix} 1.00000 & 0.49996 & 0.57995 \\ 0.64003 & 0.36192 & 0.39409 \\ 0.53007 & 0.23399 & 0.38996 \end{bmatrix} \quad \text{and}$$

$$P = \begin{bmatrix} 0.53998 & 0.32195 & 0.31394 \\ 0.41996 & 0.26505 & 0.23782 \\ 0.28103 & 0.13379 & 0.20222 \end{bmatrix} . \tag{72}$$

Estimates of the latent matrices from P* and P as shown by

$$L_1 = \begin{bmatrix} 1.00000 & 0.10130 & 0.10075 \\ 1.00000 & 0.70075 & 0.89937 \\ 1.00000 & 0.90087 & 0.19970 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1.00000 & 0.30123 & 0.30033 \\ 1.00000 & 0.40010 & 0.79975 \\ 1.00000 & 0.80122 & 0.39954 \end{bmatrix}$$

(73)

$$\hat{N} = \begin{bmatrix} 0.20093 & -0.00000 & 0.00000 \\ -0.00000 & 0.50059 & -0.00000 \\ -0.00000 & -0.00000 & 0.29847 \end{bmatrix}$$

$$D = \begin{bmatrix} 0.10192 & 0 & 0 \\ 0 & 0.50028 & 0 \\ 0 & 0 & 0.90147 \end{bmatrix}$$

(72) are given in (73). Comparison of these estimates with those in (70) reveals a very close correspondence. Again, checking the accuracy of computations results in a perfect reproduction of P; $L_1^T\hat{N}DL_2$ is represented in (74).

$$L_1^T\hat{N}DL_2 = \begin{bmatrix} 0.53998 & 0.32195 & 0.31394 \\ 0.41996 & 0.26505 & 0.23782 \\ 0.28103 & 0.13379 & 0.20222 \end{bmatrix} \qquad (74)$$

A more significant change will now be made in Π^* and Π; elements in these matrices are to be varied randomly in the fourth decimal place. The new manifest matrices are established as

$$P^* = \begin{bmatrix} 1.00000 & 0.49960 & 0.57950 \\ 0.64030 & 0.36120 & 0.39490 \\ 0.53070 & 0.23390 & 0.38960 \end{bmatrix} \quad \text{and}$$

$$P = \begin{bmatrix} 0.53980 & 0.32150 & 0.31340 \\ 0.41960 & 0.26550 & 0.23800 \\ 0.28130 & 0.13370 & 0.20240 \end{bmatrix} . \tag{75}$$

L_1, L_2, $\hat{N}$, and D as computed from the current P* and P data matrices in (75) are given in (77). The differences between these estimates and those in (70) have become greater in magnitude with error introduced in the fourth decimal place. However, all estimates remain close to the true values. Finally, a perfect check on the computational accuracy results as is shown by (76).

$$L_1^T\hat{N}DL_2 = \begin{bmatrix} 0.53980 & 0.32150 & 0.31340 \\ 0.41960 & 0.26550 & 0.23800 \\ 0.28130 & 0.13370 & 0.20240 \end{bmatrix} \tag{76}$$

$$
L_1 = \begin{bmatrix} 1.00000 & 0.11291 & 0.10777 \\ 1.00000 & 0.70750 & 0.89403 \\ 1.00000 & 0.90895 & 0.19712 \end{bmatrix}
$$

$$
L_2 = \begin{bmatrix} 1.00000 & 0.31219 & 0.30332 \\ 1.00000 & 0.40097 & 0.79774 \\ 1.00000 & 0.81243 & 0.39541 \end{bmatrix}
$$

$$
\hat{N} = \begin{bmatrix} 0.20955 & -0.00000 & -0.00000 \\ -0.00000 & 0.50552 & -0.00000 \\ 0.00000 & -0.00000 & 0.28492 \end{bmatrix} \tag{77}
$$

$$
D = \begin{bmatrix} 0.11901 & 0 & 0 \\ 0 & 0.50275 & 0 \\ 0 & 0 & 0.91501 \end{bmatrix}
$$

All of the results previously presented concerning the sensitivity of the latent class model with $m = 3$ are summarized in Tables 2, 3, and 4 in a form similar to that used by Anderson for reporting estimated structures. Arbitrarily grouping the five items necessary for the solution according to the specifications of this model, we have items 1 and 2 in the first set, items 3 and 4 in the second set, and item 5 as the stratifier.

Table 2

Latent Probability Estimates from Manifest Matrices
(error-free)

Classes	1	2	3
Proportions	0.20000	0.50000	0.30000
Item 1	0.10000	0.70000	0.90000
Item 2	0.10000	0.90000	0.20001
Item 3	0.30000	0.40000	0.80000
Item 4	0.30000	0.80000	0.40000
Item 5	0.10000	0.50000	0.90000

Table 3

Latent Probability Estimates from Manifest Matrices
(random error introduced in the fifth decimal)

Classes	1	2	3
Proportions	0.20093	0.50059	0.29847
Item 1	0.10130	0.70075	0.90087
Item 2	0.10075	0.89937	0.19970
Item 3	0.30123	0.40010	0.80122
Item 4	0.30033	0.79975	0.39954
Item 5	0.10192	0.50028	0.90147

Table 4

Latent Probability Estimates from Manifest Matrices
(random error introduced in the fourth decimal)

Classes	1	2	3
Proportions	0.20955	0.50552	0.28492
Item 1	0.11291	0.70750	0.90895
Item 2	0.10777	0.89403	0.19712
Item 3	0.31219	0.40097	0.81243
Item 4	0.30332	0.79774	0.39541
Item 5	0.11901	0.50275	0.91501

2. Sampling from a Theoretical Population

With reference to the previous introduction of error into the manifest matrices, we must consider the possibility that the random alteration of the manifest parameters may not have been consistent with the nature of the data. The elements of Π^* and Π are probabilities of positive response to single items and joint probabilities of positive response to more than one item. Since these will be related, one might question the appropriateness of the changes made and therefore the validity of the results.

The ideal method for testing the sensitivity of the latent class model to erroneous p_i, p_{ij}, and p_{ijk} values would be to simulate the actual sampling situation. To this end, a population of respondents has been generated for which $m = 3$, $N = 100{,}000$, and, as before, latent parameters correspond to those used by Anderson. The characteristics of each class are given in Tables 5, 6, and 7. Twenty samples of 1,000 subjects were randomly drawn from this theoretical population. Analysis of the data resulted in only fourteen acceptable solutions.[13] These structures are summarized in Tables 8-21; by comparing these with the estimated probabilities in Table 2, it is apparent that the accuracy of estimation has decreased from that in the previous demonstration. Even with samples

Table 5

Characteristics of First Theoretical Latent Class

Probability of Positive Response

Item 1	0.9	Item 3	0.8	Item 5	0.9
Item 2	0.2	Item 4	0.4		

Response Pattern[12]	Probability of Occurrence	Frequency in Class	Cumulative Frequency
00000	0.00096	29	29
00001	0.00864	259	288
00010	0.00064	19	307
00011	0.00576	173	480
00100	0.00384	115	595
00101	0.03456	1037	1632
00110	0.00256	77	1709
00111	0.02304	691	2400
01000	0.00024	7	2407
01001	0.00216	65	2472
01010	0.00016	5	2477
01011	0.00144	43	2520
01100	0.00096	29	2549
01101	0.00864	259	2808
01110	0.00064	19	2827
01111	0.00576	173	3000
10000	0.00864	259	3259
10001	0.07776	2333	5592
10010	0.00576	173	5765
10011	0.05184	1555	7320
10100	0.03456	1037	8357
10101	0.31104	9331	17688
10110	0.02304	691	18379
10111	0.20736	6221	24600
11000	0.00216	65	24665
11001	0.01944	583	25248
11010	0.00144	43	25291
11011	0.01296	389	25680
11100	0.00864	259	25939
11101	0.07776	2333	28272
11110	0.00576	173	28445
11111	0.05184	1555	30000
	1.00000	30000	

Table 6

Characteristics of Second Theoretical Latent Class

Probability of Positive Response

Item 1	0.7	Item 3	0.4	Item 5	0.5
Item 2	0.9	Item 4	0.8		

Response[12] Pattern	Probability of Occurrence	Frequency in Class	Cumulative Frequency
00000	0.00180	90	30090
00001	0.00180	90	30180
00010	0.00720	360	30540
00011	0.00720	360	30900
00100	0.00120	60	30960
00101	0.00120	60	31020
00110	0.00480	240	31260
00111	0.00480	240	31500
01000	0.01620	810	32310
01001	0.01620	810	33120
01010	0.06480	3240	36360
01011	0.06480	3240	39600
01100	0.01080	540	40140
01101	0.01080	540	40680
01110	0.04320	2160	42840
01111	0.04320	2160	45000
10000	0.00420	210	45210
10001	0.00420	210	45420
10010	0.01680	840	46260
10011	0.01680	840	47100
10100	0.00280	140	47240
10101	0.00280	140	47380
10110	0.01120	560	47940
10111	0.01120	560	48500
11000	0.03780	1890	50390
11001	0.03780	1890	52280
11010	0.15120	7560	59840
11011	0.15120	7560	67400
11100	0.02520	1260	68660
11101	0.02520	1260	69920
11110	0.10080	5040	74960
11111	0.10080	5040	80000
	1.00000	50000	

Table 7

Characteristics of Third Theoretical Latent Class

Probability of Positive Response

Item 1	0.1	Item 3	0.3	Item 5	0.1
Item 2	0.1	Item 4	0.3		

Response Pattern[12]	Probability of Occurrence	Frequency in Class	Cumulative Frequency
00000	0.35721	7144	87144
00001	0.03969	794	87938
00010	0.15309	3062	91000
00011	0.01701	340	91340
00100	0.15309	3062	94402
00101	0.01701	340	94742
00110	0.06561	1312	96054
00111	0.00729	146	96200
01000	0.03969	794	96994
01001	0.00441	88	97082
01010	0.01701	340	97422
01011	0.00189	38	97460
01100	0.01701	340	97800
01101	0.00189	38	97838
01110	0.00729	146	97984
01111	0.00081	16	98000
10000	0.03969	794	98794
10001	0.00441	88	98882
10010	0.01701	340	99222
10011	0.00189	38	99260
10100	0.01701	340	99600
10101	0.00189	38	99638
10110	0.00729	146	99784
10111	0.00081	16	99800
11000	0.00441	88	99888
11001	0.00049	10	99898
11010	0.00189	38	99936
11011	0.00021	4	99940
11100	0.00189	38	99978
11101	0.00021	4	99982
11110	0.00081	16	99998
11111	0.00009	2	100000
	1.00000	20000	

of 1,000 subjects, Π^* and Π have been distorted more than when changes were made in the fourth and fifth decimals of the entries in these matrices. Thus, the solution is at least moderately affected by sampling error. In order to completely cover the implications of these results, however, discussion must be delayed until the next chapter where the characteristics of the solution will be examined in depth.

Table 8

Latent Probability Estimates from Manifest Matrices (sample #1)

Classes	1	2	3
Proportions	0.25051	0.49930	0.25020
Item 1	0.22279	0.69094	0.94462
Item 2	0.05784	0.94220	0.17501
Item 3	0.34056	0.35871	0.83588
Item 4	0.31611	0.79835	0.36436
Item 5	0.16738	0.58903	0.91032

Table 9

Latent Probability Estimates from Manifest Matrices (sample #2)

Classes	1	2	3
Proportions	0.20865	0.46393	0.32742
Item 1	0.18225	0.66844	0.90667
Item 2	0.12021	0.93936	0.32717
Item 3	0.36257	0.38568	0.74651
Item 4	0.32205	0.78807	0.45262
Item 5	0.06177	0.50522	0.97343

Table 10

Latent Probability Estimates from Manifest Matrices (sample #3)

Classes	1	2	3
Proportions	0.18197	0.42183	0.39621
Item 1	0.09098	0.73922	0.85433
Item 2	0.13281	0.83976	0.29696
Item 3	0.29563	0.27966	0.77426
Item 4	0.24362	0.90087	0.43442
Item 5	0.06476	0.46113	0.82974

Table 11

Latent Probability Estimates from Manifest Matrices (sample #5)

Classes	1	2	3
Proportions	0.19300	0.48528	0.32173
Item 1	0.05548	0.68052	0.91573
Item 2	0.13469	0.89786	0.26121
Item 3	0.21257	0.41923	0.75913
Item 4	0.27272	0.85847	0.40982
Item 5	0.07594	0.46449	0.87734

Table 12

Latent Probability Estimates from Manifest Matrices (sample #6)

Classes	1	2	3
Proportions	0.20363	0.49196	0.30442
Item 1	0.12190	0.73633	0.78392
Item 2	0.12877	0.90856	0.13397
Item 3	0.27262	0.40009	0.79987
Item 4	0.24929	0.81133	0.31233
Item 5	0.05693	0.50249	0.86145

Table 13

Latent Probability Estimates from Manifest Matrices (sample #8)

Classes	1	2	3
Proportions	0.23676	0.37282	0.39043
Item 1	0.20838	0.72637	0.77572
Item 2	0.27874	0.88494	0.27171
Item 3	0.21724	0.36404	0.77824
Item 4	0.32601	0.95532	0.37050
Item 5	0.04869	0.53248	0.83999

Table 14

Latent Probability Estimates from Manifest Matrices (sample #9)

Classes	1	2	3
Proportions	0.20870	0.42491	0.36640
Item 1	0.17913	0.66500	0.89134
Item 2	0.24063	0.89583	0.23709
Item 3	0.32162	0.38871	0.81243
Item 4	0.24496	0.89196	0.44632
Item 5	0.10741	0.56493	0.89417

Table 15

Latent Probability Estimates from Manifest Matrices (sample #10)

Classes	1	2	3
Proportions	0.25059	0.45629	0.29313
Item 1	0.07434	0.68837	0.97527
Item 2	0.11245	0.89335	0.27152
Item 3	0.40173	0.36980	0.84572
Item 4	0.30603	0.86251	0.37809
Item 5	0.21666	0.51571	0.84486

Table 16

Latent Probability Estimates from Manifest Matrices (sample #11)

Classes	1	2	3
Proportions	0.18183	0.57094	0.24723
Item 1	0.00830	0.65074	0.96248
Item 2	0.00511	0.88838	0.10055
Item 3	0.29089	0.40058	0.82676
Item 4	0.27932	0.74728	0.39056
Item 5	0.17782	0.47576	0.86979

Table 17

Latent Probability Estimates from Manifest Matrices (sample #12)

Classes	1	2	3
Proportions	0.21980	0.43411	0.34609
Item 1	0.17094	0.72512	0.91085
Item 2	0.16550	0.88971	0.27616
Item 3	0.24941	0.29794	0.88124
Item 4	0.27668	0.87312	0.40359
Item 5	0.13602	0.50690	0.84544

Table 18

Latent Probability Estimates from Manifest Matrices (sample #14)

Classes	1	2	3
Proportions	0.18583	0.51772	0.29646
Item 1	0.08332	0.67027	0.84559
Item 2	0.09983	0.88037	0.20709
Item 3	0.28201	0.39635	0.76176
Item 4	0.19960	0.81797	0.36751
Item 5	0.04124	0.47219	0.90331

Table 19

Latent Probability Estimates from Manifest Matrices (sample #17)

Classes	1	2	3
Proportions	0.15493	0.62560	0.21948
Item 1	0.11884	0.71096	0.87807
Item 2	0.12784	0.81018	0.10242
Item 3	0.30295	0.47691	0.83757
Item 4	0.29637	0.75115	0.33045
Item 5	0.02746	0.54181	0.94277

Table 20

Latent Probability Estimates from Manifest Matrices (sample #18)

Classes	1	2	3
Proportions	0.19736	0.40348	0.39916
Item 1	0.00829	0.65871	0.91336
Item 2	0.12723	0.90987	0.31259
Item 3	0.27494	0.34835	0.78712
Item 4	0.26857	0.90858	0.41437
Item 5	0.14736	0.47123	0.78861

Table 21

Latent Probability Estimates from Manifest Matrices (sample #19)

Classes	1	2	3
Proportions	0.28327	0.44652	0.27022
Item 1	0.24439	0.70831	0.90899
Item 2	0.17531	0.85297	0.22408
Item 3	0.30520	0.39052	0.93733
Item 4	0.25222	0.88870	0.31039
Item 5	0.20317	0.48838	0.86659

CHAPTER IV

THE PROPERTIES OF THE METHOD

To promote a greater understanding of the results given in the previous chapter, some characteristics of the computational method must be examined. With reference to the data necessary for the solution, items to be analyzed are selected in two sets of m-1 items plus one additional item, m being equal to the number of latent classes hypothesized. Assuming that m = 3, we will consider five items arbitrarily numbered 1, 2, 3, 4, and 5. The total number of different ways of analyzing is equal to 120; if we consistently define item 5 as the stratifier, the total number of groupings or analyses is reduced to 24. The pairs of manifest data matrices (P* and P) associated with the different ways of choosing items for analysis are presented in Tables 22, 23, and 24. Theoretically, if we analyzed each of these twenty-four pairs of matrices, twenty-four identical estimates of the latent parameters would result. However, since sampling will introduce varying amounts of error into each element of the data matrices, this will not be the case. Different analyses will then most likely give different estimates of the latent probabilities. The solution might therefore include the use of all twenty-four

Table 22

Manifest Data Matrices - Group 1

Analysis	P*	P
1	$\begin{bmatrix} 1 & p_3 & p_4 \\ p_1 & p_{13} & p_{14} \\ p_2 & p_{23} & p_{24} \end{bmatrix}$	$\begin{bmatrix} p_5 & p_{35} & p_{45} \\ p_{15} & p_{135} & p_{145} \\ p_{25} & p_{235} & p_{245} \end{bmatrix}$
2	$\begin{bmatrix} 1 & p_4 & p_3 \\ p_1 & p_{14} & p_{13} \\ p_2 & p_{24} & p_{23} \end{bmatrix}$	$\begin{bmatrix} p_5 & p_{45} & p_{35} \\ p_{15} & p_{145} & p_{135} \\ p_{25} & p_{245} & p_{235} \end{bmatrix}$
3	$\begin{bmatrix} 1 & p_3 & p_4 \\ p_2 & p_{23} & p_{24} \\ p_1 & p_{13} & p_{14} \end{bmatrix}$	$\begin{bmatrix} p_5 & p_{35} & p_{45} \\ p_{25} & p_{235} & p_{245} \\ p_{15} & p_{135} & p_{145} \end{bmatrix}$
4	$\begin{bmatrix} 1 & p_4 & p_3 \\ p_2 & p_{24} & p_{23} \\ p_1 & p_{14} & p_{13} \end{bmatrix}$	$\begin{bmatrix} p_5 & p_{45} & p_{35} \\ p_{25} & p_{245} & p_{235} \\ p_{15} & p_{145} & p_{135} \end{bmatrix}$
5	$\begin{bmatrix} 1 & p_1 & p_2 \\ p_3 & p_{13} & p_{23} \\ p_4 & p_{14} & p_{24} \end{bmatrix}$	$\begin{bmatrix} p_5 & p_{15} & p_{25} \\ p_{35} & p_{135} & p_{235} \\ p_{45} & p_{145} & p_{245} \end{bmatrix}$
6	$\begin{bmatrix} 1 & p_2 & p_1 \\ p_3 & p_{23} & p_{13} \\ p_4 & p_{24} & p_{14} \end{bmatrix}$	$\begin{bmatrix} p_5 & p_{25} & p_{15} \\ p_{35} & p_{235} & p_{135} \\ p_{45} & p_{245} & p_{145} \end{bmatrix}$
7	$\begin{bmatrix} 1 & p_1 & p_2 \\ p_4 & p_{14} & p_{24} \\ p_3 & p_{13} & p_{23} \end{bmatrix}$	$\begin{bmatrix} p_5 & p_{15} & p_{25} \\ p_{45} & p_{145} & p_{245} \\ p_{35} & p_{135} & p_{235} \end{bmatrix}$
8	$\begin{bmatrix} 1 & p_2 & p_1 \\ p_4 & p_{24} & p_{14} \\ p_3 & p_{23} & p_{13} \end{bmatrix}$	$\begin{bmatrix} p_5 & p_{25} & p_{15} \\ p_{45} & p_{245} & p_{145} \\ p_{35} & p_{235} & p_{135} \end{bmatrix}$

Table 23

Manifest Data Matrices - Group 2

Analysis	P*	P
9	$\begin{bmatrix} 1 & P_2 & P_4 \\ P_1 & P_{12} & P_{14} \\ P_3 & P_{23} & P_{34} \end{bmatrix}$	$\begin{bmatrix} P_5 & P_{25} & P_{45} \\ P_{15} & P_{125} & P_{145} \\ P_{35} & P_{235} & P_{345} \end{bmatrix}$
10	$\begin{bmatrix} 1 & P_4 & P_2 \\ P_1 & P_{14} & P_{12} \\ P_3 & P_{34} & P_{23} \end{bmatrix}$	$\begin{bmatrix} P_5 & P_{45} & P_{25} \\ P_{15} & P_{145} & P_{125} \\ P_{35} & P_{345} & P_{235} \end{bmatrix}$
11	$\begin{bmatrix} 1 & P_2 & P_4 \\ P_3 & P_{23} & P_{34} \\ P_1 & P_{12} & P_{14} \end{bmatrix}$	$\begin{bmatrix} P_5 & P_{25} & P_{45} \\ P_{35} & P_{235} & P_{345} \\ P_{15} & P_{125} & P_{145} \end{bmatrix}$
12	$\begin{bmatrix} 1 & P_4 & P_2 \\ P_3 & P_{34} & P_{23} \\ P_1 & P_{14} & P_{12} \end{bmatrix}$	$\begin{bmatrix} P_5 & P_{45} & P_{25} \\ P_{35} & P_{345} & P_{235} \\ P_{15} & P_{145} & P_{125} \end{bmatrix}$
13	$\begin{bmatrix} 1 & P_1 & P_3 \\ P_2 & P_{12} & P_{23} \\ P_4 & P_{14} & P_{34} \end{bmatrix}$	$\begin{bmatrix} P_5 & P_{15} & P_{35} \\ P_{25} & P_{125} & P_{235} \\ P_{45} & P_{145} & P_{345} \end{bmatrix}$
14	$\begin{bmatrix} 1 & P_3 & P_1 \\ P_2 & P_{23} & P_{12} \\ P_4 & P_{34} & P_{14} \end{bmatrix}$	$\begin{bmatrix} P_5 & P_{35} & P_{15} \\ P_{25} & P_{235} & P_{125} \\ P_{45} & P_{345} & P_{145} \end{bmatrix}$
15	$\begin{bmatrix} 1 & P_1 & P_3 \\ P_4 & P_{14} & P_{34} \\ P_2 & P_{12} & P_{23} \end{bmatrix}$	$\begin{bmatrix} P_5 & P_{15} & P_{35} \\ P_{45} & P_{145} & P_{345} \\ P_{25} & P_{125} & P_{235} \end{bmatrix}$
16	$\begin{bmatrix} 1 & P_3 & P_1 \\ P_4 & P_{34} & P_{14} \\ P_2 & P_{23} & P_{12} \end{bmatrix}$	$\begin{bmatrix} P_5 & P_{35} & P_{15} \\ P_{45} & P_{345} & P_{145} \\ P_{25} & P_{235} & P_{125} \end{bmatrix}$

Table 24

Manifest Data Matrices - Group 3

Analysis	P*	P
17	$\begin{bmatrix} 1 & p_2 & p_3 \\ p_1 & p_{12} & p_{13} \\ p_4 & p_{24} & p_{34} \end{bmatrix}$	$\begin{bmatrix} p_5 & p_{25} & p_{35} \\ p_{15} & p_{125} & p_{135} \\ p_{45} & p_{245} & p_{345} \end{bmatrix}$
18	$\begin{bmatrix} 1 & p_3 & p_2 \\ p_1 & p_{13} & p_{12} \\ p_4 & p_{34} & p_{24} \end{bmatrix}$	$\begin{bmatrix} p_5 & p_{35} & p_{25} \\ p_{15} & p_{135} & p_{125} \\ p_{45} & p_{345} & p_{245} \end{bmatrix}$
19	$\begin{bmatrix} 1 & p_2 & p_3 \\ p_4 & p_{24} & p_{34} \\ p_1 & p_{12} & p_{13} \end{bmatrix}$	$\begin{bmatrix} p_5 & p_{25} & p_{35} \\ p_{45} & p_{245} & p_{345} \\ p_{15} & p_{125} & p_{135} \end{bmatrix}$
20	$\begin{bmatrix} 1 & p_3 & p_2 \\ p_4 & p_{34} & p_{24} \\ p_1 & p_{13} & p_{12} \end{bmatrix}$	$\begin{bmatrix} p_5 & p_{35} & p_{25} \\ p_{45} & p_{345} & p_{245} \\ p_{15} & p_{135} & p_{125} \end{bmatrix}$
21	$\begin{bmatrix} 1 & p_1 & p_4 \\ p_2 & p_{12} & p_{24} \\ p_3 & p_{13} & p_{34} \end{bmatrix}$	$\begin{bmatrix} p_5 & p_{15} & p_{45} \\ p_{25} & p_{125} & p_{245} \\ p_{35} & p_{135} & p_{345} \end{bmatrix}$
22	$\begin{bmatrix} 1 & p_4 & p_1 \\ p_2 & p_{24} & p_{12} \\ p_3 & p_{34} & p_{13} \end{bmatrix}$	$\begin{bmatrix} p_5 & p_{45} & p_{15} \\ p_{25} & p_{245} & p_{125} \\ p_{35} & p_{345} & p_{135} \end{bmatrix}$
23	$\begin{bmatrix} 1 & p_1 & p_4 \\ p_3 & p_{13} & p_{34} \\ p_2 & p_{12} & p_{24} \end{bmatrix}$	$\begin{bmatrix} p_5 & p_{15} & p_{45} \\ p_{35} & p_{135} & p_{345} \\ p_{25} & p_{125} & p_{245} \end{bmatrix}$
24	$\begin{bmatrix} 1 & p_4 & p_1 \\ p_3 & p_{34} & p_{13} \\ p_2 & p_{24} & p_{12} \end{bmatrix}$	$\begin{bmatrix} p_5 & p_{45} & p_{15} \\ p_{35} & p_{345} & p_{135} \\ p_{25} & p_{245} & p_{125} \end{bmatrix}$

pairs of matrices and the computation of an average of the results to arrive at overall estimates of the latent parameters. But note that the matrices in Tables 22, 23, and 24 are classified as three different groups; these subdivisions correspond to analyses 1 to 8, 9 to 16, and 17 to 24 in that order. Each pair of matrices in one specific group consists of elements common to all other pairs of matrices in that group. The only difference between the matrix pairs in each group is the positioning of their elements. Thus, the $P^{*-1}P$ products formed from each pair of matrices within a group will be similarity transforms of one another. Where Q is a nonsingular matrix, we have

$$
\begin{aligned}
P^*_s &= P^*Q \quad , \\
P_s &= PQ \quad , \text{ and} \\
P^{*-1}_s &= Q^{-1}P^{*-1} \quad .
\end{aligned}
\tag{78}
$$

From (78), the product matrix for which the latent roots are to be determined is defined as shown in (79) where $P^{*-1}_s P_s$ is a matrix similar to an original matrix $P^{*-1}P$. Since similar matrices have the same characteristic polynomial and therefore the same roots, the estimated matrix D will be identical

$$P_s^{*-1}P_s = Q^{-1}P^{*-1}PQ = Q^{-1}[P^{*-1}P]Q \tag{79}$$

for each pair of matrices in the same group. Although the latent vectors will be different due to the fact that the structures of the $P^{*-1}P$ product matrices are not alike, the latent parameter estimates from the manifest matrix pairs in the same group will still be identical after taking into account a repositioning of the elements in L_1 and L_2. Both of these matrices have m columns, the first consisting of 1s and the next m-1 containing latent probability estimates for that many items over m latent classes. Remembering that two sets of m-1 items were selected (plus a stratifier which is of no concern at the moment), it could be shown that the item parameter estimates in L_1 are for the items included in the first set and that those in L_2 are for the items in the second set. As an example with m = 3, suppose items 4 and 1 were taken as the first set and items 3 and 2 as the second; then the second and third columns of L_1 would contain estimates of the latent probabilities for items 4 and 1 respectively and the corresponding columns of L_2, estimates for items 3 and 2 in that order. Thus, the structure of the estimated latent matrices is dependent upon the initial item selection.

Since we have three groups of eight similar matrices when m = 3, more than one solution using matrices from the same group would be wasteful because identical estimates would result. Referring to the twenty-four possible analyses, only three are necessary for the purpose of estimating the latent parameters when using a constant stratifying item. Considering the five different stratifiers, we have a total of fifteen relevant analyses. Thus, knowledge of the similarity transforms can be used to choose from all possible analyses only those needed for efficient estimation; the only pairs of matrices which should be used are those containing elements absent from matrices which have undergone previous analysis.

With reference to the results presented in Tables 8-21 in the preceding chapter, it will be recalled that the manifest probability estimates from each of twenty random samples were subjected to fifteen analyses. For six of these samples, none of the fifteen analyses resulted in an estimated structure; for each of the remaining fourteen samples, no more than four structures were obtained. These results are presented in the Appendix. When more than one structure was computed for a given sample, estimates were averaged to give a single composite structure for that sample. In such a case, the variation between corresponding latent parameter estimates for a specific item was generally around 0.05 but, in some instances, was as great as 0.25. Twenty samples of 500 observations were also analyzed but only six of these resulted in at least one estimated structure. Thus, it was found that samples of 1,000 were necessary for consistent estimation of the latent probabilities.

Erroneous estimates might be due to three possible causes. The first is inappropriate specification of the number of latent classes; this is of no concern in the present research since the theoretical population which was generated for testing the model had a known number of classes. The second detrimental factor is that of latent matrix singularity. If one or more of the latent matrices is singular, inaccurate estimation will occur because we cannot state that the diagonal entries of D and the latent roots of $P^{*-1}P$ are equivalent. Hence, the assumption that all the latent matrices be of rank m is crucial. Of course, singularity is not a yes-no characteristic but instead exists in various amounts or degrees. From experience with this procedure to date, it can be generally stated that a violation of this assumption may be recognized when $L_1^T\hat{N}DL_2 \neq P$, the number of discrepant elements possibly being related to the degree of singularity. The trace of $\hat{N}$ may also be used as an indicator of this property although this does not seem to be as sensitive as the test previously mentioned. If tr $\hat{N} \neq 1.00000$, singularity in one or more of the latent matrices should be suspected. At any rate, good estimates were obtained when $|\Lambda_2|$ was equal to +0.12. Reference is made to the Appendix and the second estimated structure for sample #1 where items 3 and 5 were chosen as the second set. It should be mentioned, however, that no estimates emerged when items 2 and 4 were in the same set, the determinant being of the magnitude ±0.03. Since all other determinants were greater than 0.12 (and less than 0.60) absolute, it does not seem likely that singularity of the latent matrices caused poor estimation of the latent parameters except possibly in the case cited above. The combination of items 2 and 4 in a set occurs in only three of the fifteen possible analyses for each sample. Therefore, any error in the estimated structures must, in most instances, be attributed to the third cause, that of sampling error. The conclusion to be drawn is that the procedure proposed by Anderson is very sensitive to random error; samples of 1,000 observations have been shown to be necessary for consistent approximation of the latent parameters.

The last property of the solution is that the introduction of error into the manifest matrices does not affect the check on the accuracy of computations; the matrix $L_1^T\hat{N}DL_2$ gave a perfect reproduction of P in each case regardless of erroneous manifest probabilities. Since (60) and (61) are identities, a solution will take place even if the estimates are beyond the limits of probabilities. As a result, the structure may not always be interpretable from the psychological standpoint.[13]

CHAPTER V

CONCLUSIONS

It would seem from the foregoing evidence that the computational method proposed by Anderson for determining latent probabilities is far too sensitive to sampling error to be of any practical value. The necessity of using large samples prohibits its use in almost all situations. However, this fact might not constitute sufficient evidence for a definite rejection of this method if this was its only defect. From the previous discussion of the characteristics of the solution, we are left with the impression that the technique is, to say the least, inherently awkward. First of all, the number of classes must be determined by a trial-and-error procedure; this difficulty might not be apparent in the example given in Chapter III because the number of different latent classes contributing to the manifest probabilities was known to be three. Without previous knowledge, however, many analyses might be run before the true number of classes becomes known. In addition, the row and column interchanges in the latent matrices caused by different patterns of item selection will almost always lead to ambiguous estimates in the practical situation where the theoretical structure is not known. This will be especially true when the latent probabilities associated with the stratifier over two or more latent classes are approximately equal. Finally, Anderson's solution does not make use of all the data simultaneously; the result is that many different structures could be computed.

All of these considerations force a reevaluation of Green's method. Although this technique does require the estimation of several elements in the manifest data matrix, it does not require initial knowledge of the number of latent classes; furthermore, it uses all the data at the same time with the result that only one solution is computed. Even though some of the original data has to be approximated, this is also a tolerated characteristic of factor analysis. Assuming the computer is utilized, the increased computational complexity of Green's solution is a minor difficulty. These facts create a need for further research in this area. A similar investigation of Green's procedure might lead to results more encouraging than those presented in this paper. At any rate, practical application will be delayed until a more robust model is derived and empirically investigated.

APPENDIX

ESTIMATED STRUCTURES FOR FOURTEEN SAMPLES

Computer output for the fourteen samples with acceptable[13] structures is given on the following pages. With reference to the theoretical structure used by Anderson [1954] as covered in Section 1 of Chapter III, the latent item probabilities may be arbitrarily numbered 1 through 5 in the first class, 6 through 10 in the second, and 11 through 15 in the third. By an appropriate numbering of the estimates in the following structures, the row and column interchanges in the latent matrices become apparent. Where more than one structure was computed for a given sample, the corresponding estimates of each were averaged.

SAMPLE #1

ANALYSIS NUMBER *P12345*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.25747	0.51468	0.22785
ITEM 1	(11) 0.24436	(6) 0.69460	(1) 0.95497
ITEM 2	(12) 0.04040	(7) 0.93297	(2) 0.16859
ITEM 3	(13) 0.33969	(8) 0.36976	(3) 0.85686
ITEM 4	(14) 0.31494	(9) 0.78916	(4) 0.35876
ITEM 5	(15) 0.17495	(10) 0.58680	(5) 0.94772

ANALYSIS NUMBER *P14352*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(1) 2	(2) 3
PROPORTIONS	0.21347	0.28728	0.49926
ITEM 1	(11) 0.13966	(1) 0.93432	(6) 0.68057
ITEM 4	(14) 0.30455	(4) 0.35149	(9) 0.80723
ITEM 3	(13) 0.33154	(3) 0.77330	(8) 0.36069
ITEM 5	(15) 0.14533	(5) 0.83159	(10) 0.58703
ITEM 2	(12) 0.04734	(2) 0.19816	(7) 0.92531

ANALYSIS NUMBER *P23451*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.26355	0.48865	0.24780
ITEM 2	(12) 0.05951	(7) 0.97739	(2) 0.14413
ITEM 3	(13) 0.34926	(8) 0.33983	(3) 0.86720
ITEM 4	(14) 0.33007	(9) 0.78923	(4) 0.38884
ITEM 5	(15) 0.16641	(10) 0.59573	(5) 0.92026
ITEM 1	(11) 0.25185	(6) 0.69421	(1) 0.93786

ANALYSIS NUMBER *P25341*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.26756	0.49459	0.23785
ITEM 2	(12) 0.08409	(7) 0.93311	(2) 0.18916
ITEM 5	(15) 0.18283	(10) 0.58654	(5) 0.94170
ITEM 3	(13) 0.34175	(8) 0.36454	(3) 0.84617
ITEM 4	(14) 0.31489	(9) 0.80777	(4) 0.35834
ITEM 1	(11) 0.25529	(6) 0.69436	(1) 0.95131

SAMPLE #2

****

ANALYSIS NUMBER *P23451*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.20865	0.46393	0.32742
ITEM 2	(12) 0.12021	(7) 0.93936	(2) 0.32717
ITEM 3	(13) 0.36257	(8) 0.38568	(3) 0.74651
ITEM 4	(14) 0.32205	(9) 0.78807	(4) 0.45262
ITEM 5	(15) 0.06177	(10) 0.50522	(5) 0.97343
ITEM 1	(11) 0.18225	(6) 0.66844	(1) 0.90667

****

SAMPLE #3

****

ANALYSIS NUMBER *P12345*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.16507	0.48616	0.34877
ITEM 1	(11) 0.06632	(6) 0.72699	(1) 0.86481
ITEM 2	(12) 0.08272	(7) 0.86308	(2) 0.19999
ITEM 3	(13) 0.28242	(8) 0.36130	(3) 0.75331
ITEM 4	(14) 0.22481	(9) 0.80206	(4) 0.46438
ITEM 5	(15) 0.06004	(10) 0.43729	(5) 0.88452

****

ANALYSIS NUMBER *P25341*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.19886	0.35749	0.44365
ITEM 2	(12) 0.18289	(7) 0.81643	(2) 0.39393
ITEM 5	(15) 0.06948	(10) 0.48497	(5) 0.77496
ITEM 3	(13) 0.30884	(8) 0.19801	(3) 0.79521
ITEM 4	(14) 0.26243	(9) 0.99968	(4) 0.40446
ITEM 1	(11) 0.11563	(6) 0.75144	(1) 0.84384

****

SAMPLE #5

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ANALYSIS NUMBER *P12345*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.19450	0.47506	0.33045
ITEM 1	(11) 0.09031	(6) 0.65963	(1) 0.92322
ITEM 2	(12) 0.16165	(7) 0.94880	(2) 0.19315
ITEM 3	(13) 0.21275	(8) 0.42194	(3) 0.75406
ITEM 4	(14) 0.29813	(9) 0.83499	(4) 0.44287
ITEM 5	(15) 0.05841	(10) 0.47667	(5) 0.85095

****

ANALYSIS NUMBER *P14235*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.19223	0.54735	0.26042
ITEM 1	(11) 0.00468	(6) 0.72173	(1) 0.92182
ITEM 4	(14) 0.20707	(9) 0.83825	(4) 0.39312
ITEM 2	(12) 0.16998	(7) 0.84318	(2) 0.19894
ITEM 3	(13) 0.21428	(8) 0.44590	(3) 0.79004
ITEM 5	(15) 0.09171	(10) 0.47713	(5) 0.92239

****

ANALYSIS NUMBER *P23451*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.18408	0.51593	0.30000
ITEM 2	(12) 0.08730	(7) 0.85574	(2) 0.29477
ITEM 3	(13) 0.17817	(8) 0.46585	(3) 0.72620
ITEM 4	(14) 0.27425	(9) 0.85452	(4) 0.36549
ITEM 5	(15) 0.10225	(10) 0.42125	(5) 0.94282
ITEM 1	(11) 0.05098	(6) 0.67540	(1) 0.92720

****

ANALYSIS NUMBER *P25341*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.20117	0.40278	0.39604
ITEM 2	(12) 0.11984	(7) 0.94373	(2) 0.35797
ITEM 5	(15) 0.05140	(10) 0.48292	(5) 0.79321
ITEM 3	(13) 0.24507	(8) 0.34321	(3) 0.76622
ITEM 4	(14) 0.31144	(9) 0.90610	(4) 0.43779
ITEM 1	(11) 0.07594	(6) 0.66531	(1) 0.89068

****

SAMPLE #6

****

ANALYSIS NUMBER *P12345*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.21315	0.45069	0.33616
ITEM 1	(11) 0.14905	(6) 0.73453	(1) 0.78292
ITEM 2	(12) 0.20571	(7) 0.92378	(2) 0.16307
ITEM 3	(13) 0.27316	(8) 0.39609	(3) 0.76827
ITEM 4	(14) 0.24763	(9) 0.84599	(4) 0.32705
ITEM 5	(15) 0.06010	(10) 0.49438	(5) 0.84894

****

ANALYSIS NUMBER *P25341*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.19410	0.53322	0.27267
ITEM 2	(12) 0.05182	(7) 0.89333	(2) 0.10487
ITEM 5	(15) 0.05375	(10) 0.51060	(5) 0.87396
ITEM 3	(13) 0.27208	(8) 0.40409	(3) 0.83147
ITEM 4	(14) 0.25094	(9) 0.77667	(4) 0.29761
ITEM 1	(11) 0.09475	(6) 0.73812	(1) 0.78491

****

SAMPLE #8

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ANALYSIS NUMBER *P12453*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.23676	0.37282	0.39043
ITEM 1	(11) 0.20838	(6) 0.72637	(1) 0.77572
ITEM 2	(12) 0.27874	(7) 0.88494	(2) 0.27171
ITEM 4	(14) 0.32601	(9) 0.95532	(4) 0.37050
ITEM 5	(15) 0.04869	(10) 0.53248	(5) 0.83999
ITEM 3	(13) 0.21724	(8) 0.36404	(3) 0.77824

****

SAMPLE #9

ANALYSIS NUMBER *P12345*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.18730	0.43210	0.38060
ITEM 1	(11) 0.11896	(6) 0.66975	(1) 0.88366
ITEM 2	(12) 0.15453	(7) 0.92966	(2) 0.24789
ITEM 3	(13) 0.31990	(8) 0.38430	(3) 0.79882
ITEM 4	(14) 0.28111	(9) 0.84239	(4) 0.46072
ITEM 5	(15) 0.13477	(10) 0.51036	(5) 0.90182

ANALYSIS NUMBER *P12453*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.23872	0.42303	0.33825
ITEM 1	(11) 0.30166	(6) 0.62925	(1) 0.91587
ITEM 2	(12) 0.41866	(7) 0.85564	(2) 0.18654
ITEM 4	(14) 0.14191	(9) 0.95992	(4) 0.44951
ITEM 5	(15) 0.05351	(10) 0.64004	(5) 0.90308
ITEM 3	(13) 0.31235	(8) 0.43394	(3) 0.80374

ANALYSIS NUMBER *P25341*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.20007	0.41959	0.38034
ITEM 2	(12) 0.14871	(7) 0.90219	(2) 0.30683
ITEM 5	(15) 0.13394	(10) 0.54438	(5) 0.87760
ITEM 3	(13) 0.33262	(8) 0.34788	(3) 0.83474
ITEM 4	(14) 0.31186	(9) 0.87357	(4) 0.42873
ITEM 1	(11) 0.11676	(6) 0.69601	(1) 0.87448

SAMPLE #10

ANALYSIS NUMBER *P23451*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.25714	0.48235	0.26051
ITEM 2	(12) 0.07581	(7) 0.92171	(2) 0.20696
ITEM 3	(13) 0.38701	(8) 0.42065	(3) 0.83139
ITEM 4	(14) 0.31944	(9) 0.81199	(4) 0.40381
ITEM 5	(15) 0.22797	(10) 0.51269	(5) 0.88320
ITEM 1	(11) 0.09930	(6) 0.68873	(1) 0.99903

ANALYSIS NUMBER *P25341*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.24403	0.43023	0.32575
ITEM 2	(12) 0.14909	(7) 0.86499	(2) 0.33608
ITEM 5	(15) 0.20534	(10) 0.51873	(5) 0.80652
ITEM 3	(13) 0.41644	(8) 0.31895	(3) 0.86005
ITEM 4	(14) 0.29261	(9) 0.91303	(4) 0.35237
ITEM 1	(11) 0.04937	(6) 0.68801	(1) 0.95151

SAMPLE #11

****

ANALYSIS NUMBER *P25341*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.18183	0.57094	0.24723
ITEM 2	(12) 0.00511	(7) 0.88838	(2) 0.10055
ITEM 5	(15) 0.17782	(10) 0.47576	(5) 0.86979
ITEM 3	(13) 0.29089	(8) 0.40058	(3) 0.82676
ITEM 4	(14) 0.27932	(9) 0.74728	(4) 0.39056
ITEM 1	(11) 0.00830	(6) 0.65074	(1) 0.96248

****

SAMPLE #12

ANALYSIS NUMBER *P12345*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.20716	0.42100	0.37183
ITEM 1	(11) 0.16387	(6) 0.70125	(1) 0.91123
ITEM 2	(12) 0.14723	(7) 0.91626	(2) 0.27365
ITEM 3	(13) 0.24696	(8) 0.29913	(3) 0.83614
ITEM 4	(14) 0.25244	(9) 0.86353	(4) 0.44416
ITEM 5	(15) 0.13354	(10) 0.47648	(5) 0.84645

ANALYSIS NUMBER *P25341*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.23243	0.44721	0.32035
ITEM 2	(12) 0.18377	(7) 0.86315	(2) 0.27867
ITEM 5	(15) 0.13850	(10) 0.53731	(5) 0.84442
ITEM 3	(13) 0.25185	(8) 0.29674	(3) 0.92634
ITEM 4	(14) 0.30092	(9) 0.88271	(4) 0.36302
ITEM 1	(11) 0.17800	(6) 0.74898	(1) 0.91046

SAMPLE #14

ANALYSIS NUMBER *P14235*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.17912	0.61679	0.20409
ITEM 1	(11) 0.07706	(6) 0.68166	(1) 0.87095
ITEM 4	(14) 0.18783	(9) 0.80963	(4) 0.23509
ITEM 2	(12) 0.11464	(7) 0.76711	(2) 0.16816
ITEM 3	(13) 0.26161	(8) 0.45355	(3) 0.78591
ITEM 5	(15) 0.03911	(10) 0.49024	(5) 0.99768

ANALYSIS NUMBER *P23451*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.19253	0.41865	0.38882
ITEM 2	(12) 0.08501	(7) 0.99362	(2) 0.24602
ITEM 3	(13) 0.30241	(8) 0.33915	(3) 0.73760
ITEM 4	(14) 0.21137	(9) 0.82630	(4) 0.49992
ITEM 5	(15) 0.04336	(10) 0.45414	(5) 0.80893
ITEM 1	(11) 0.08957	(6) 0.65887	(1) 0.82023

SAMPLE #17

****

ANALYSIS NUMBER *P12345*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.16285	0.60367	0.23348
ITEM 1	(11) 0.13963	(6) 0.71101	(1) 0.87396
ITEM 2	(12) 0.18340	(7) 0.81007	(2) 0.13328
ITEM 3	(13) 0.30617	(8) 0.47483	(3) 0.82448
ITEM 4	(14) 0.29664	(9) 0.76663	(4) 0.32938
ITEM 5	(15) 0.03654	(10) 0.54190	(5) 0.92908

****

ANALYSIS NUMBER *P25341*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.14700	0.64752	0.20548
ITEM 2	(12) 0.07227	(7) 0.81028	(2) 0.07156
ITEM 5	(15) 0.01837	(10) 0.54171	(5) 0.95646
ITEM 3	(13) 0.29973	(8) 0.47898	(3) 0.85065
ITEM 4	(14) 0.29609	(9) 0.73566	(4) 0.33152
ITEM 1	(11) 0.09804	(6) 0.71090	(1) 0.88218

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SAMPLE #18

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ANALYSIS NUMBER *P25341*
E S T I M A T E D   S T R U C T U R E
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CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.19736	0.40348	0.39916
ITEM 2	(12) 0.12723	(7) 0.90987	(2) 0.31259
ITEM 5	(15) 0.14736	(10) 0.47123	(5) 0.78861
ITEM 3	(13) 0.27494	(8) 0.34835	(3) 0.78712
ITEM 4	(14) 0.26857	(9) 0.90858	(4) 0.41437
ITEM 1	(11) 0.00829	(6) 0.65871	(1) 0.91336

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SAMPLE #19

ANALYSIS NUMBER *P12345*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.18824	0.52979	0.28197
ITEM 1	(11) 0.13916	(6) 0.67198	(1) 0.90008
ITEM 2	(12) 0.07872	(7) 0.77953	(2) 0.21703
ITEM 3	(13) 0.28796	(8) 0.37564	(3) 0.92485
ITEM 4	(14) 0.15586	(9) 0.83759	(4) 0.29051
ITEM 5	(15) 0.13741	(10) 0.46389	(5) 0.85601

ANALYSIS NUMBER *P15234*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(1) 2	(2) 3
PROPORTIONS	0.34889	0.26187	0.38924
ITEM 1	(11) 0.34547	(1) 0.93823	(6) 0.69308
ITEM 5	(15) 0.29723	(5) 0.89182	(10) 0.45154
ITEM 2	(12) 0.18560	(2) 0.30520	(7) 0.88460
ITEM 3	(13) 0.33802	(3) 0.94631	(8) 0.38090
ITEM 4	(14) 0.27410	(4) 0.38294	(9) 0.92252

ANALYSIS NUMBER *P12453*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.38193	0.32698	0.29109
ITEM 1	(11) 0.32685	(6) 0.76830	(1) 0.89301
ITEM 2	(12) 0.29810	(7) 0.98422	(2) 0.18318
ITEM 4	(14) 0.37473	(9) 0.94640	(4) 0.35187
ITEM 5	(15) 0.23135	(10) 0.54165	(5) 0.85036
ITEM 3	(13) 0.30581	(8) 0.41394	(3) 0.89955

ANALYSIS NUMBER *P25341*

E S T I M A T E D   S T R U C T U R E

CLASSES	(3) 1	(2) 2	(1) 3
PROPORTIONS	0.21401	0.54006	0.24593
ITEM 2	(12) 0.13882	(7) 0.76351	(2) 0.19092
ITEM 5	(15) 0.14668	(10) 0.49642	(5) 0.86818
ITEM 3	(13) 0.28901	(8) 0.39158	(3) 0.97861
ITEM 4	(14) 0.20419	(9) 0.84829	(4) 0.21624
ITEM 1	(11) 0.16609	(6) 0.69989	(1) 0.90463

FOOTNOTES

[1]Equation (1) is from Anderson [1954] and equation (3), from Lazarsfeld [1959]. Without going into detail here, $\phi(x)$ is assumed to be discrete in (1) but continuous in (3).

[2]Referring to (3), Lazarsfeld notes that a_i^0 and a_i^1 correspond to the probability that a person will give a positive response to item i regardless of his position on continuum x and the discriminating power of item i respectively.

[3]This is commented on by Horst [1959], pp. 38-39.

[4]The order of presentation is similar to that used by Horst in that the excessive steps in the solution of Green have been eliminated.

[5]From (22), a more basic form of (24) which might be used is $Q = B^{-1}P_{(1)}[B^T]^{-1}$.

[6]The notation of Anderson will be used here and for the remainder of this paper. It is independent of Green's notation in that the same symbol may have a different meaning in each solution. For instance, Anderson uses p as a statistic while Green uses it as a parameter.

[7]Summation concerns α where $\alpha = 1, \ldots, m$.

[8]One characteristic of the programmed solution is that the real positive latent roots are found in an order related to their magnitude, that is, from smallest to largest. This is inherent in the operations performed by MATVEC, the subroutine used to compute the roots of the matrix $P^{*-1}P$.

[9]The maximum number of latent classes to which the program LSA can be applied is, at present, equal to ten.

[10]The matrices Π^* and Π as computed by THEO are theoretical in that they are based entirely on Λ_1, Λ_2, N, and Δ.

[11]Slightly inaccurate computation is evidenced by the fact that, although the first column of both Λ_1 and Λ_2 consisted of 1s and the off-diagonal elements of N were 0s, the respective entries in the computed estimates are 1.00000 and either 0.00000 or -0.00000. This means that, where five zeroes are present to the right of the decimal point, there is a fractional part of the number which is less than the absolute magnitude 5×10^{-6}, this amount of error being considered negligible.

[12]A positive response to one of the five items is represented as a "1" and a negative response, as a "0".

[13]The only structures to be considered were those in which all estimates were between 0 and +1.

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